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CONTENTS

PAGE

Acoustic Impedance of a Porous Plate . . . .
Dr. Arvind Mohan Srivastava 1

Choice of X-Ray Tube Window Materials . . . .
Chintamani Mande 6

Statistical Correlation of Ultrasonic Studies in Different Gels—I . . .
Dr. Arvind Mohan Srivastava and Anand Kumar Srivastava 22

Generalization of a Theorem of Newsom . . . .
Nirmala Pandey 35

On the Singularities of a Class of Laplace-Abel Integral . . . .
Nirmala Pandey 42

On the Convergence of Generalised Laplacestieltjes Integrals . . .
Snehlata 51

An Extension of Hadamard's Multiplication Theorem. . . . .
Nirmala Pandey 56

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ACOUSTIC IMPEDANCE OF A POROUS PLATE

BY DR. ARVIND MOHAN SRIVASTAVA

(Physics Department, Allahabad University)

Received February 1, 1952

INTRODUCTION

THE measurement of acoustic impedance of different sound absorbing materials is of fundamental importance in the design of studios for sound reproduction and recording, architectural acoustics and in various diverse fields concerning sound insulation and absorption. In this laboratory Prof. Ghosh started the study of acoustic impedance of various materials having a large porosity. Upon his initiation Chandra Kanta,[2] Srivastava[3] and in recent years Chatterji[4] developed different techniques to study acoustic impedance of usual absorbers like cellotex, felt, cork padding and other Indian woollen pads (Nâmda). The method of Chandra Kanta depends upon the determination of air load on the electrical impedance of sound source. To counteract the lack of high precision in her measurements a modified apparatus on the lines of that devised by Beranek[5] was used by Chatterji.

The present investigation is a theoretical attempt to obtain an expression for acoustic impedance which can be of assistance to the experimental experts. The theory can be applied to such porous substances as cotton-wool, felt, celotex, etc., which have large voids as shown by Table I taken from Chatterji.[6]

TABLE I

Sample	Volume V_1	Calculated Air Volume V_2	Solid Volume V_3	Percentage Porosity $\frac{V_1 - V_3}{V_2}$
Celotex 1 ..	51·5	71·3	6·7	87
Felt	26·3	74·6	3·4	87
Celotex 2 ..	45·0	71·3	6·7	85

INTRODUCTORY THEORY

The problem is related to finding the impedance of a gelatinous substance in which the preponderance of a liquid phase over the spherical solids is quite well known. It was in such a search that this paper was undertaken and can be applied to other porous materials equally well.

Let us consider a rectangular chamber of dimension $2l \times m \times n$ along the axes of x, y, z respectively. Let us imagine a wall drawn of the thin porous slab at $x = l$ and the volume of chamber is reduced to lmn, *i.e.*, half that of the whole chamber. In accordance with the usual procedure impedance at the sample partition can be obtained by looking into the tube of length l and cross-section mn terminated by an impedance of the x-wall at $x = 0$. In the presence of the sample let this impedance be modified from Z_2 to Z_3 and it is assumed that the presence of screen does not disturb the particle velocity or pressure distributions. Taking the impedance of screen to be Z_1 we get,

$$Z_3 = Z_2 + Z_1 \tag{1}$$

The usual equation of motion for the propagation of sound waves of velocity c, excess pressure p and in air of density ρ is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{2}$$

according to Morse[7] the pressure distribution in a room having one standing wave is given by,

$$p_m = P_x \, P_y \, P_z \tag{3}$$

where,

$$P_x = \cosh\left[\left(\frac{\pi X}{l}\right)\left(-\eta_x + j\delta_x\right) + f(x)\right] \tag{4}$$

The frequencies of normal modes of vibration are given by,

$$f_m = \frac{c}{2}\left[\frac{\delta_x^2 - \eta_x^2}{l^2} + \frac{\delta_y^2 - \eta_y^2}{m^2} + \frac{\delta_z^2 - \eta_z^2}{n^2}\right]^{\frac{1}{2}} \tag{5}$$

The propagation constants k_m are given by,

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In which k_m^2 (= $(2\pi f_m)^2$. δ_x, η_x and $f(x)$ are constants whose values depend upon boundary conditions of (4). δ_x and η_x depend only upon the impedance of x-walls and similarly δ_y, η_y and δ_z, η_z on those of y and z walls.

Acoustic Impedance

At the plane $x = l$ the acoustical effect will be a maximum because the particle velocities are maximum here (at the anti-nodal point) hence from (6) the value of propagation constant k_3 in the presence of screen is,

$$k_3 = \frac{\pi c^2}{2f_3}\left[\frac{\delta_{x3} \eta_{x3}}{l^2} + \frac{\delta_y \eta_y}{m^2} + \frac{\delta_z \eta_z}{n^2}\right] \tag{8}$$

and in its absence is,

$$k_2 = \frac{\pi c^2}{2f_2}\left[\frac{\delta_x \eta_x}{l^2} + \frac{\delta_y \eta_y}{m^2} + \frac{\delta_z \eta_z}{n^2}\right] \tag{9}$$

In case $f_2 = f_3 = f$ (say) then from (8) and (9) we have,

$$(\delta_{x3}\eta_{x3} - \delta_x \eta_x) = (k_3 - k_2)\frac{2fl^2}{\pi c^2} \tag{10}$$

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From the above two we have,

$$(\delta_x^2 - \eta_x^2) - (\delta_{x3}^2 - \eta_{x3}^2) = (f_2^2 - f_3^2)\frac{4l^2}{c^2} \tag{12}$$

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when there is one pressure nodal plane at $x = l$. Applying the condition that the materials are very porous as per Table I. the Morse treatment would give the constants $\delta_x^2 - \eta_x^2$ and $\delta_x \eta_x$ in terms of the impedances of x-walls.

The impedance $\rho c \gamma_3\ e^{j\psi_3}$ of $x = 0$ wall is very large due to its rigidity while that of $x = l$ wall $\rho c \gamma_2\, e^{j\psi_2}$ is small by comparison due to a pressure node at $x = l$. With these conditions one can get the series solution of Morse and then,

$$(\gamma_3 \cos \psi_3 - \gamma_2 \cos \psi_2) = (k_3 - k_2)\ \frac{l}{c} \tag{10'}$$

and,

$$(\gamma_3 \sin \psi_3 - \gamma_2 \sin \psi_2) = 4\,(f_2^2 - f_3^2)\ \frac{\pi l^2}{c^2} \tag{12'}$$

where $Z_3 = \rho c \gamma_3\, e^{j\psi_3}$; $Z_2 = \rho c \gamma_2\, e^{j\psi_2}$ are the values of impedances with and without screen. The screen impedance,

$$Z = R + jX \tag{13}$$

$$= Z_3 - Z_2$$

Since,

$$Z = \rho c\,(\gamma \cos \psi + j\gamma \sin \psi) \tag{14}$$

the real and imaginary terms of sample impedance are,

$$\left.\begin{aligned} R &= \rho\,(k_3 - k_2)\, l \\ X &= 4\rho\,(f_2^2 - f_3^2)\ \frac{\pi l^2}{c} \end{aligned}\right\} \tag{15}$$

if $k^2 << (2\pi f)^2$, $\frac{f_2}{f_3} \simeq 1$ and the porosity of screen being large the wave formation is not impaired.

Thus a construction of rectangular chamber of dimensions $2l \times m \times n$ will enable a determination of acoustic impedance of a sample placed at $x = l$ provided measurements are made at the lowest resonant frequency of the chamber. In case P_2/P_3 is the ratio of pressure in a bare chamber to that in a sample using chamber then Harris[8] has shown that an alternate expression for R can be

$$R = \rho k_2 \left(\frac{P_2}{P_3} - 1\right) l.$$

By an application of the above theory easy determination of acoustic impedance of highly porous bodies is possible provided they are of the form of a slab and place inside a rectangular chamber dividing it in two halves.

ACKNOWLEDGEMENT

The author is indebted to the late Prof. R. N. Ghosh, D.SC., F.N.I., F.A.S. (America), for his initiation of the problem and to Shri N. N. Ghosh for giving access to consult his father's library.

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CHOICE OF X-RAY TUBE WINDOW MATERIALS*

By Chintamani Mande

(Department of Physics, Allahabad University)

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(Communicated by Dr. G. B. Deodhar, Allahabad University)

Introduction

The absorption of X-rays traversing through matter differs from substance to substance. Thus X-rays are much more strongly absorbed by some substances than by others. In the selection of the material for X-ray tube windows, it is necessary to choose such substances in which the absorption of X-rays is comparatively small, to minimise the loss of the radiation intensity as far as possible.

In demountable tubes of the Hadding, Shearer, Siegbahn, etc., types used usually in X-ray spectroscopy, the windows selected are generally of aluminium, cellophane and mica. In the sealed tubes of the Coolidge type, the windows are made of special glasses in which the absorption of X-rays is small. It appears worthwhile to have an accurate and exact knowledge of the extent of the absorption of X-rays through these window materials. This paper compares the X-ray absorption coefficients of aluminium, cellophane, six commonly occurring varieties of mica and six varieties of special window glasses, calculated theoretically for some wave-lengths in the region 0·5 to 1·5 A.U. The wave-lengths chosen are the $K\alpha_1$ lines of 29 Cu, 30 Zn, 32 Ge, 34 Se, 37 Rb, 42 Mo and 47 Ag, substances very often used for anti-cathodes in X-ray tubes.

General Theory of Calculations

It has been established that the absorption of X-rays occurs by two processes. The first is the photo-electric or true absorption in which the energy of the beam is absorbed due to the ejection of the photo-electrons. Here the energy of the incident radiation is transferred into the kinetic energy of the ejected electron and the potential energy of the excited atom. The second process is of scattering in which the X-ray energy is spent in producing scattered rays. Under this process scattering of both the types, with and without change of wave-length are grouped. If we represent

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τ as the absorption due to the ejection of photo-electrons and σ as the absorption due to scattering, the total linear absorption may be written as,

$$\mu = \tau + \sigma \tag{1}$$

In terms of the mass absorption coefficient we may write,

$$\frac{\mu}{\rho} = \frac{\tau}{\rho} + \frac{\sigma}{\rho} \tag{2}$$

where ρ is the density of the absorbing material.

The absorption due to scattering is very well expressed by the well-known Klein-Nishina[1] formula, according to which the scattering coefficient per electron is given by,

$$\sigma_e = \frac{8}{3}\frac{\pi e^4}{m^2c^4} \cdot \frac{3}{4}\left[\frac{1+\alpha_0}{\alpha_0^2}\left\{\frac{2(1+\alpha_0)}{1+2\alpha_0} - \frac{1}{\alpha_0}\log(1+2\alpha_0)\right\} + \frac{1}{2\alpha_0}\log(1+2\alpha_0) - \frac{1+3\alpha_0}{(1+2\alpha_0)^2}\right] \tag{3}$$

in which e and m are the charge and mass of the electron, c the velocity of light $\alpha_0 = \frac{h\nu}{mc^2}$ and ν is incident frequency. Thus we have the mass scattering coefficient, if Z be the atomic number of the element under consideration and A its atomic weight, and N the Avagadro number, as

$$\frac{\sigma}{\rho} = \frac{8}{3}\frac{\pi e^4}{m^2c^4} \cdot \frac{N}{A} \cdot Z \cdot \frac{3}{4}\left[\frac{1+\alpha_0}{\alpha_0^2}\left\{\frac{2(1+\alpha_0)}{1+2\alpha_0} - \frac{1}{\alpha_0}\log(1+2\alpha_0)\right\} + \frac{1}{2\alpha_0}\log(1+2\alpha_0) - \frac{1+3\alpha_0}{(1+2\alpha_0)^2}\right] \tag{4}$$

A large number of empirical and theoretical expressions have been given by different workers for the true absorption of X-rays. Most of these formulæ have many limitations and they do not give values in good comparison with the experimental data. Recently, however, Victoreen[2] has shown that the photo-electric atomic absorption coefficient τ_a of any element is given with excellent accuracy by the expression,

$$\tau_a = \frac{C}{\nu^3} - \frac{D}{\nu^4}, \tag{5}$$

where C and D are constants which change value between absorption discontinuities and ν is the incident frequency. The constants C and D are given by,

$$C = \nu_1\nu_2 (3\pi e^2/mc),$$

$$D = \nu_1\nu_2\nu_3 (3\pi e^2/mc),$$

where ν_1, ν_2 and ν_3 are critical frequencies obtained from the accepted formula of spectral emission,

$$\nu_x = Rc\ (Z-s)^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) + Rc\left[\alpha^2\ (Z-s)^4\left(\frac{1}{n_1^4}-\frac{1}{n_2^4}\right)\left(\frac{n}{k}-\frac{3}{4}\right)+\cdots\right] \tag{6}$$

in which R is Rydberg's number, c the velocity of light, Z is atomic number, α is Sommerfeld's fine structure constant, s is a screening constant and n_1 and n_2 are quantum numbers. For wave-lengths less than K absorption discontinuities, Victoreen finds that satisfactory agreement in calculation is obtained by assigning the following values: $s = 0, \frac{1}{2}, 1$, etc., and $n = 1, 2, 3, 4$, etc., $\alpha = (2\pi e^2/ch)$ and $k = n$.

From equation (5) we may write for the mass photo-electric absorption coefficient,

$$\frac{\tau}{\rho} = \left(\frac{\nu_1\nu_2}{\nu_3} - \frac{\nu_1\nu_2\nu_3}{\nu_4}\right) \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A} \tag{7}$$

Adding the right-hand terms of equations (4) and (7) we get,

$$\left(\frac{\mu}{\rho}\right) = \left[\left(\frac{\nu_1\nu_2}{\nu_3} - \frac{\nu_1\nu_2\nu_3}{\nu_4}\right) \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A}\right]$$
$$+\left[\frac{8}{3}\frac{\pi e^4}{m^2c^4} \cdot \frac{N}{A} \cdot Z \cdot \frac{3}{4}\left(\frac{1+\alpha_0}{\alpha_0^2}\left\{\frac{2(1+\alpha_0)}{1+2\alpha_0} - \frac{1}{\alpha_0}\log(1+2\alpha_0)\right\}\right.\right.$$
$$\left.\left.+\frac{1}{2\alpha_0}\log(1+2\alpha_0) - \frac{1+3\alpha_0}{(1+2\alpha_0)^2}\right)\right]$$
$$= \left(\nu_1\nu_2 \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A}\right)\lambda^3 - \left(\nu_1\nu_2\nu_3 \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A}\right)\lambda^4 + \sigma_e N\frac{Z}{A}, \tag{8}$$

putting λ for the wave-length of the incident radiation.

For simplicity of calculation we may write the above equation as,

$$\left(\frac{\mu}{\rho}\right) = P\lambda^3 - Q\lambda^4 + \sigma e N\frac{Z}{A}, \tag{9}$$

where

$$P = \left(\nu_1\nu_2 \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A}\right)$$

and

$$Q = \left(\nu_1 \nu_2 \nu_3 \cdot \frac{3\pi e^2}{mc} \cdot \frac{N}{A}\right).$$

The values of the constants P, Q and Z/A may be found in Victoreen's paper referred to above.

The mass absorption coefficients of the elements required in this work, namely, H, Li, Be, B, C, O, F, Na, Al, Si, K and Fe for the $K\alpha_1$ lines of 29 Cu, 30 Zn, 32 Ge, 37 Rb, 42 Mo and 47 Ag, as calculated from equation (9), have been given below in Table I.

TABLE I

Element	Wave-length in A.U.						
	AgK_1 0·5583	MoK_1 0·7078	RbK_1 0·9236	SeK_1 1·1025	GeK_1 1·2513	ZnK_1 1·4322	CuK_1 1·5374
1 H	0·3703	0·3796	0·3900	0·4001	0·4100	0·4254	0·4352
3 Li	0·1865	0·2165	0·2832	0·3670	0·4605	0·6073	0·7114
4 Be	0·2278	0·2964	0·4555	0·6568	0·8813	1·235	1·487
5 B	0·2769	0·3895	0·6534	0·9877	1·361	1·951	2·368
6 C	0·3971	0·6194	1·146	1·812	2·557	3·732	4·723
8 O	0·7321	1·300	2·648	4·356	6·260	9·256	11·378
9 F	0·9338	1·718	3·580	5·937	8·561	12·684	15·594
11 Na	1·625	3·114	6·640	11·087	16·022	27·749	29·202
12 Mg	2·145	4·157	8·911	14·896	21·521	31·878	39·171
13 Al	2·638	5·149	11·080	18·521	26·742	39·561	48·571
14 Si	3·288	6·451	13·911	23·263	33·576	49·642	60·921
19 K	8·131	16·076	34·505	57·205	81·792	119·454	145·486
26 Fe	19·656	38·304	79·858	128·806	179·302	253·014	301·595

The mass absorption coefficients of the elements can be easily converted into atomic absorption coefficients by multiplying them with the factor A/N. In the case of mica where exact formulæ giving their composition are available, the mass absorption coefficients of these are calculated by assuming the Additive Law[3] according to which the molecular absorption coefficient of any compound $X_x\ Y_y\ Z_z \ldots$ is given by

$$\mu_{mol} = \frac{M}{\rho N} \times \mu = x\,(\mu_a)_X + y\,(\mu_a)_Y + z\,(\mu_a)_Z + \ldots \tag{10}$$

where $(\mu_a)_X$ is atomic absorption coefficient of the element X for a given wave-length and M is the molecular weight. The mass absorption coefficient of the compound may be easily determined from the molecular

absorption coefficient. In the case of cellophane and window glasses the mass absorption coefficients are calculated from the known percentages of the various constituents.

RESULTS AND DISCUSSION

1. *Aluminium.*—The mass absorption coefficients of aluminium according to these calculations have already been given in Table I. That the calculated values show excellent agreement with the experimental results can be easily seen from Fig. 1, in which the calculated values of the mass

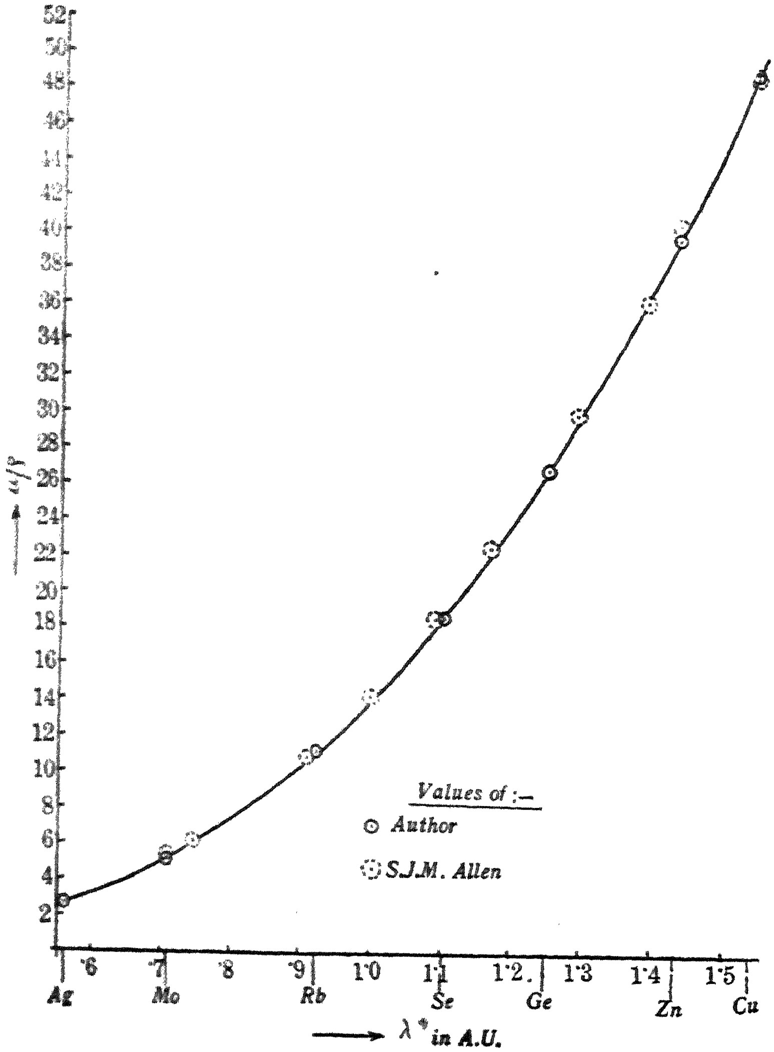

FIG. 1

absorption coefficients of aluminium have been compared with the experimental data of Allen.[1] This may also be regarded as a test of the general validity of these calculations. In Fig. 2, the logarithms of the mass absorption coefficients of aluminium have been plotted against the logarithms of

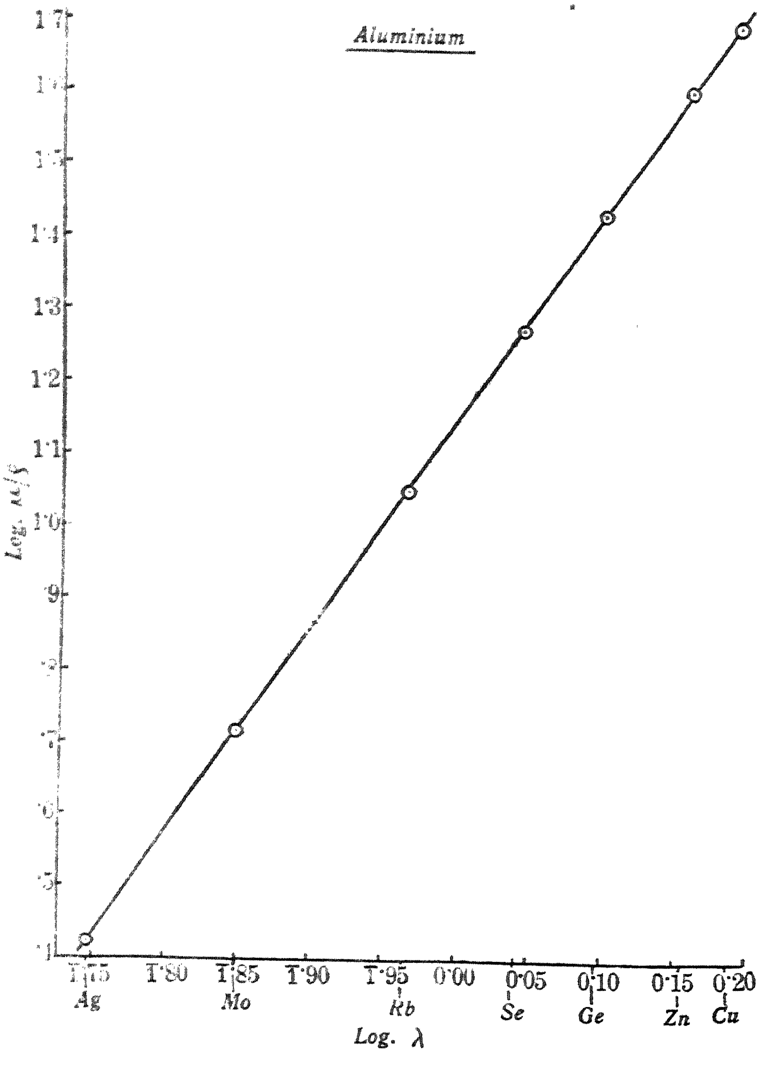

FIG. 2

wave-lengths in order to get a relationship between the mass absorption coefficients with wave-length in this region.

It is found that the mass absorption coefficient of aluminium in the region 0·5–1·5 A.U. may be roughly expressed by,

$$\frac{\mu}{\rho} = 14 \cdot 1\ \lambda^{2 \cdot 8} \tag{11}$$

Dividing the mass absorption coefficient by 2·7, the density of aluminium, its linear absorption coefficient may be easily calculated for any wave-length in this region.

2. *Mica.*—Mica exists in nature in numerous forms. The absorption of X-rays may differ widely from one variety of mica to the other. It is therefore essential to know the absorption coefficients for commonly occurring forms of mica.

Mellor[5] has given six principal forms of mica. They are given in Table II together with their constitution. Of all these micas muscovite and paragonite are the most common varieties.

TABLE II

Name			Composition
Muscovite	Potash	Mica	$H_2KAl_3(SiO_4)_3$
Paragonite	Soda	Mica	$H_2NaAl_3(SiO_4)_3$
Lepidolite	Lithia	Mica	$(HO, F)_2(Li, K)_2Al_2Si_3O_9$
Phlogopite	Magnesia	Mica	$(H, K, Mg, F)_3Mg_3Al(SiO_4)_3$
Zinnwaldite	Lithium-iron	Mica	$(HO, F)_2(Li, K)_3FeAl_3Si_5O_6$
Biotite	Magnesium-iron	Mica	$(H, K)_2(Mg, Fe)_2(Al, Fe)_2(SiO_4)_3$

The mass absorption coefficients of the various forms of mica as obtained from these calculations are shown in Table III. In Fig. 3, the

TABLE III

Radiation K_1 line of	Calculated μ/ρ values					
	Muscovite	Paragonite	Lepidolite	Zinnwaldite	Phlogopite	Biotite
Cu	42·52	31·18	48·70	83·08	51·55	132·76
Zn	34·73	25·64	39·83	68·57	42·16	110·65
Ge	23·58	17·17	27·11	47·36	28·69	77·63
Se	16·40	11·91	18·89	33·36	19·99	55·35
Rb	9·859	7·143	11·37	20·30	12·02	34·07
Mo	4·617	3·354	5·324	9·574	5·624	16·25
Ag	2·384	1·751	2·744	4·908	2·894	8·341

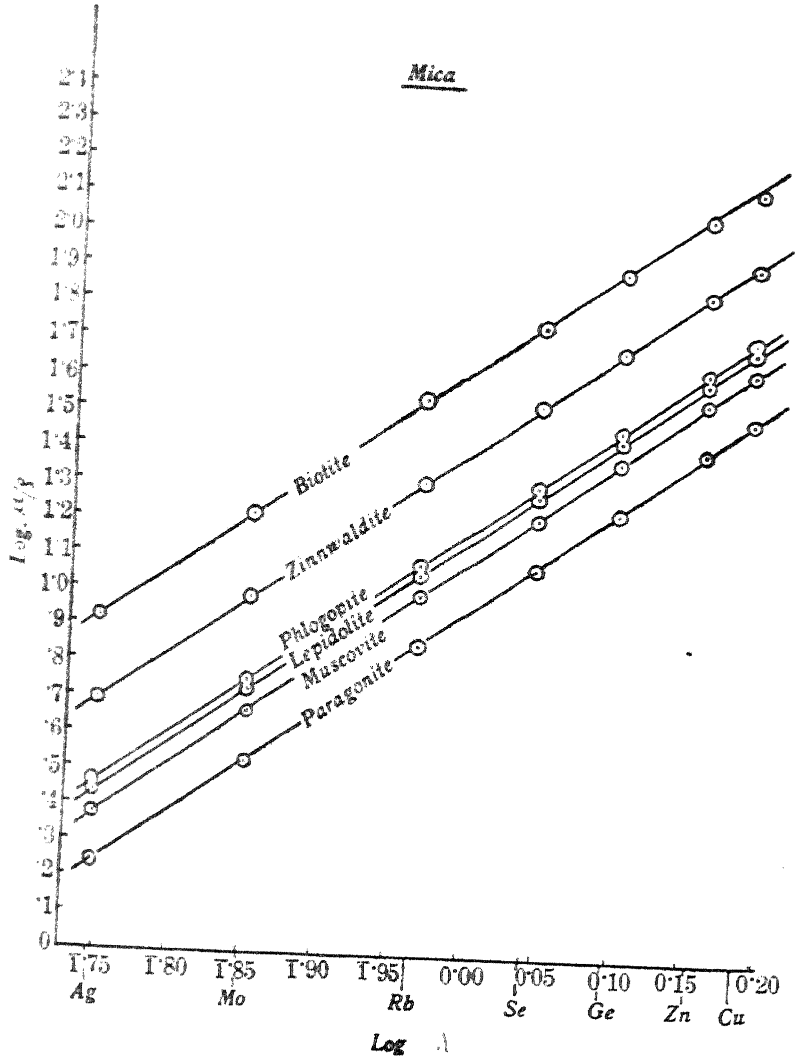

FIG. 3

logarithms of the mass absorption coefficients of the micas have been plotted against the logarithms of wave-lengths.

It is seen from Table III that the mass absorption coefficients of X-rays in the various qualities of mica differ widely from one another. The absorption coefficient is least for paragonite and has the maximum value for biotite for any wave-length. The absorption coefficient of biotite is nearly four times that of paragonite and nearly three times that of muscovite. It is therefore advisable to use paragonite and muscovite micas as a material

for X-ray tube windows, in order to reduce the absorption of X-rays in the window to a minimum.

It is interesting to compare the calculated values of the mass absorption coefficients obtained here with the experimental results of some workers.

Unnewehr[6] whose data alone have been given in the *International Critical Tables* (Vol. VI, p. 16) has measured the linear absorption coefficients of mica for the $K\alpha$ and $K\beta$ radiations of Cr, Cu, Rh, and Ag. He has not, however, mentioned the variety of mica used by him. In Table IV below, the calculated values for paragonite are compared with the experimental results of Unnewehr. The excellent agreement between columns 2 and 4 proves beyond doubt that Unnewehr had used paragonite mica in his investigations.

TABLE IV

Radiation K_1 line of	Unnewehr		Author calculated μ/ρ for paragonite
	μ	μ/ρ*	
Cu	89·2	31·84	31·18
Ag	5·1	1·82	1·75

* The densities of the micas vary from about 2·7 to 3. Here ρ is taken as 2·8, as has been done in the *I.C.T.*

Williams[7] has measured the linear absorption coefficients of mica in the region 0·4–2·3 A.U. and has shown that the linear absorption coefficient in this region can be expressed by

$$\mu = 36\cdot56\lambda^{2\cdot76} \tag{12}$$

Williams also has not mentioned the variety of mica used in his work. His results, however, agree fairly well with the calculated absorption coefficient for muscovite. This can be readily seen from Table V.

TABLE V

Radiation K_1 line of	Williams		Author calculated μ/ρ for muscovite
	μ	μ/ρ*	
Cu	119·81	42·79	42·52
Ag	7·32	2·61	2·38

It thus appears that equation (12) given by Williams is true only for muscovite mica.

From Fig. 3, it is found that the linear absorption coefficient of mica in the region 0·5–1·5 A.U. may be roughly expressed by

$$\frac{\mu}{2\cdot8} = A\,\lambda^{2\cdot8}. \tag{13}$$

where A is a constant varying with the type of mica; for muscovite $A = 12\cdot6$, for paragonite $A = 9\cdot1$, for lepidolite $A = 14\cdot5$, for phlogopite $A = 15\cdot1$, for zinnwaldite $A = 25\cdot1$ and for biotite $A = 41\cdot2$. Knowing the variety of mica, its linear absorption coefficient can be readily calculated from equation (13).

3. *Cellophane.*—This is a complex organic substance. It usually contains* 80% regenerated cellulose, 14% glycerol and 6% water. The mass absorption coefficients of cellulose, glycerol and water as obtained from these calculations are given in Table VI.

TABLE VI

Radiation K_1 line of	μ/ρ		
	Cellulose $(C_6H_{10}O_5)\,n$	Glycerol $C_3H_8O_3$	Water H_2O
Cu	7·740	7·816	10·150
Zn	6·248	6·322	8·268
Ge	4·250	4·299	5·605
Se	2·979	3·014	3·913
Rb	1·840	1·863	2·395
Mo	0·9403	0·9531	1·197
Ag	0·5607	0·5693	0·6915

From the absorption coefficients of cellulose, glycerol and water, the mass absorption coefficients of cellophane have been calculated. They are shown in Table VII.

TABLE VII

Radiation K_1 line of	Calculated μ/ρ of cellophane
Cu	7·825
Zn	6·380
Ge	4·338
Se	3·040
Rb	1·877
Mo	0·9574
Ag	0·5798

The only experimental work on the absorption coefficients of cellophane in this region appears to be that of Williams. Williams has shown that for cellophane the linear absorption coefficient may be expressed in the region 0·6–2·3 A.U., by

$$\mu = 3\cdot 52\ \lambda^{2\cdot 66}. \tag{14}$$

It will be interesting to compare some of the values of the mass absorption coefficients of cellophane obtained in this work with those of Williams. This camparison for the $K\alpha_1$ lines of Cu and Ag is shown below in Table VIII.

TABLE VIII

Radiation K_1 line of	Williams		Author calculated
	μ	μ/ρ*	μ/ρ
Cu	11·05	8·50	7·83
Ag	0·747	0·57	0·58

* The density of cellophone varies from about 1·2–1·4. Here it has been taken as 1–3.

The agreement between the values of the author and Williams appears to be fairly good though not perfect. It should, however, be remembered that the proportions of the constituent, in cellophane may vary to some extent and it may as well contain some impurities.

The logarithms of the mass absorption coefficients of cellophane have been plotted in Fig. 4 with the logarithms of wave-lengths. From Fig. 4 it appears that the linear absorption coefficient of cellophane can be given approximately by the expression

$$\frac{\mu}{1\cdot 3} = 2\cdot 5\ \lambda^{2\cdot 65} \tag{15}$$

Comparing equations (11), (13) and (15) it is at once apparent that the absorption of X-rays is much less in cellophane than in mica or aluminium for any given wave-length. It is thus more desirable to use cellophane windows in demountable X-ray tubes.

4. *Window glasses.*—Several special varieties of glasses are used for making windows in sealed X-ray tubes. It is essential that the absorption of X-rays in the window be as small as possible. For this it is necessary

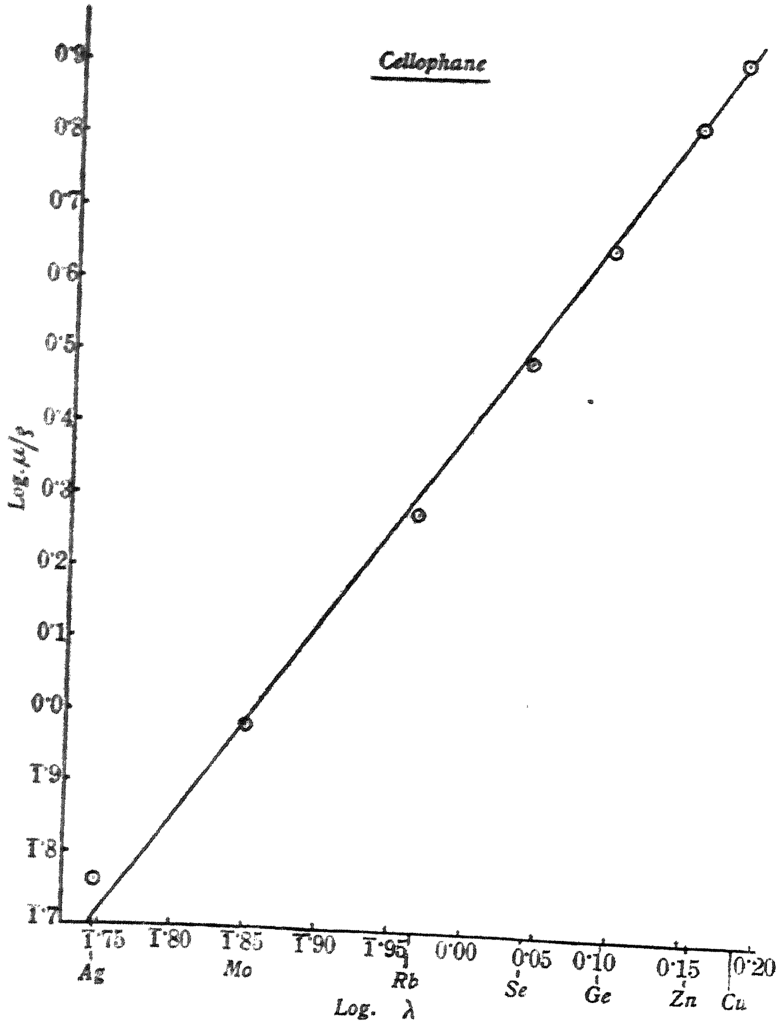

FIG. 4

and sufficient for the glass to contain only elements of low atomic number. The problem becomes difficult when it is remembered that in addition the glass must have a sufficient endurance from the chemical, mechanical and thermal point of view. Besides this, the coefficient of expansion of the window glass should be as near as possible to that of the glass of the tube.

The glasses usually used in the windows of X-ray tubes are given in Table IX together with their percentage compositions.[9]

TABLE IX

Name of the glass	Percentage composition		
	BeO	Li_2O	B_2O_3
Lindemann	2·50	14·20	83·30
Zeigler and Wellmann ..	4·40	13·60	82·00
Russian Patent 47050 I ..	13·00	15·00	72·00
Russian Patent 47050 II ..	8·50	20·30	71·20
Mazelev I	14·08	17·56	68·36
Mazelev II	11·95	17·82	70·23

The calculated absorption coefficients of the oxides constituting the glasses are given in Table X below.

TABLE X

Radiation K_1 line of	μ/ρ		
	BeO	Li_2O	B_2O_3
Cu	7·812	6·423	8·578
Zn	6·364	5·239	6·986
Ge	4·321	3·566	4·735
Se	3·022	2·503	3·309
Rb	1·858	1·550	2·028
Mo	0·9382	0·7967	1·017
Ag	0·5502	0·4786	0·5906

From the mass absorption coefficients of these oxides, the mass absorption coefficients of the window glasses have been calculated. They are given in Table XI.

TABLE XI

Radiation K_1 line of	Calculated μ/ρ values					
	Lindemann	Zeigler and Wellmann	Russ. Patent I	Russ. Patent II	Mazelev I	Mazelev II
Cu	8·253	8·251	8·155	8·075	8·092	8·102
Zn	6·722	6·721	6·663	6·578	6·592	6·600
Ge	4·559	4·558	4·506	4·463	4·471	4·477
Se	3·187	3·187	3·151	3·121	3·127	3·131
Rb	1·956	1·956	1·934	1·917	1·920	1·923
Mo	0·9838	0·9836	0·9737	0·9655	0·9672	0·9683
Ag	0·5738	0·5736	0·5685	0·5645	0·5652	0·5658

It is seen from Table XI that the mass absorption coefficients of all the window glasses for any one wave-length are nearly the same. The mass absorption coefficient is maximum for Lindemann glass and has minimum values for Mazelve glasses. From chemical, mechanical and thermal points of view also, the Mazelev glasses are considered as the best, while the original Lindemann glass, the first in this interesting glass family, appears to be the least desirable.

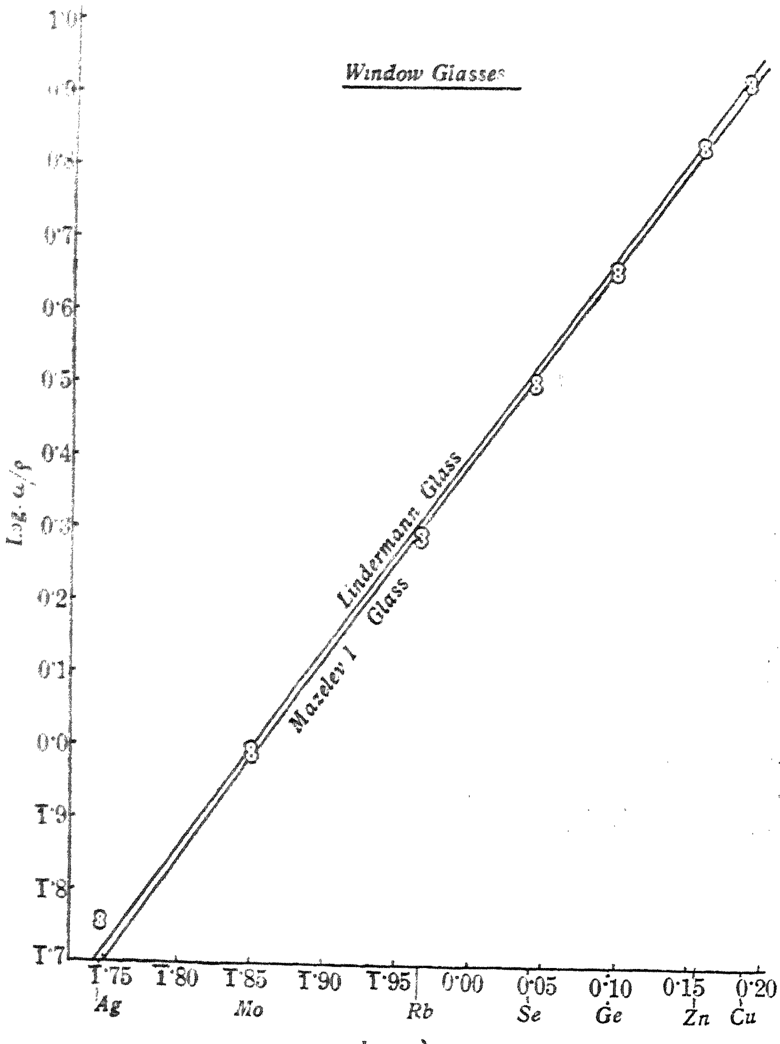

FIG. 2

It would have been interesting to compare the calculated values of the absorption coefficients of the window glasses with some experimental data, which unfortunately is not available.

The logarithms of the mass absorption coefficients of Lindemann and Mazelev glasses, the two varieties having maximum and minimum values are plotted against the logarithms of wave-lengths in Fig. 5. There is not much difference between the values of the absorption coefficients of the different varieties of the window glasses for any given wave-length and the $\log \mu/\rho - \log \lambda$ curves for the remaining varieties would fall in the same area. From Fig. 5 it appears that the absorption coefficients of the window glasses in the region 0·5–1·5 A.U. can be given roughly by following expression

$$\frac{\mu}{\rho} = 2{\cdot}5\,\lambda^{2\cdot 15}. \tag{16}$$

The density of the window glasses is approximately 1·15, from which the linear absorption coefficient of the glasses can be readily calculated.

SUMMARY

The mass absorption coefficients of aluminium, mica, cellophane and some special varieties of glasses—the substances usually used as window materials in X-ray tubes—have been calculated theoretically in the region 0·5–1·5 A.U. The wave-lengths chosen are the $K\alpha_1$ lines of 29 Cu, 30 Zn, 32 Ge, 34 Se, 37 Rb, 42 Mo and 47 Ag. The calculations are based on a recent formula given by Victoreen for the photo-electric absorption and the Klein-Nishina formula of scattering.

Aluminium, mica and cellophane windows are usually used in demountable X-ray tubes. It has been shown that the absorption of X-rays is least in cellophane for any given wave-length. It is thus desirable to use cellophane windows in demountable X-ray tubes.

In the sealed X-ray tubes, though there is not much difference in the absorption coefficients of the window glasses, the absorption in Mazelev glasses appears to be the minimum. From other points of view as well this variety appears to be superior to others.

The calculated values of the absorption coefficients of aluminium, mica and cellophane have been compared with the experimental data of some workers. The agreement in the case of aluminium and cellophane is excellent. In the case of mica, the experimental workers have not mentioned the variety of mica used by them. It has been possible to show that the mica used by

Unnewehr was paragonite mica and that used by Williams was muscovite mica. Experimental data on window glasses is not available for comparison.

Simple relationships between the absorption coefficients and wave-length have been obtained for all these substances for ready calculation of the absorption coefficients for any wave-length in the region 0·5–1·5 A.U.

ACKNOWLEDGEMENT

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STATISTICAL CORRELATION OF ULTRASONIC STUDIES IN DIFFERENT GELS—I

BY DR. ARVIND MOHAN SRIVASTAVA
(*Physics Department, Allahabad University*)
AND
SRI ANAND KUMAR SRIVASTAVA
(*Assistant Accountant-General, Allahabad*)

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ABSTRACT

The paper attempts to throw light on the basic structure of some inorganic gels inasmuch as a similarity in the behaviour of their Young's Modulus with temperature is concerned. It is observed that the variation can be represented by

$$\mathrm{Log}_e\,\eta = \log_e a_0 + a_1\mathrm{T},$$

where η is the Young's modulus, T the temperature in °C., a_0 and a_1 are two constants depending upon the nature of a particular gel sample.

Six gels of the Weimarn class are studied in which quite a good agreement is obtained regarding a_0 and a_1 as shown in Table X. Furthermore, by way of comparison Iron Silicate and Thorium Phosphate jellies have been taken. These do not give the same order of values for a_0 or a_1 which goes to show that there is a variation of property from one group to another although gels of the same group are affected to an equal and similar extent upon the application of deformation forces.

The Young's modulus is determined by the author's ultrasonic total reflection pulse technique.

1. INTRODUCTION

IN a series of recent publications originating from the work done by one of the authors (A.M.S.) the viscoelastic constants of a number of gelatinous substances falling into the Weimarn class according to Prof. Dhar's classification,[1] have been reported.[2] The dependence of these elastic modulii on such parameters as temperature, time of setting and the frequency[3] of the impressed ultrasonic waves that are employed for the purpose using the well-known pulse exploration techniques,[4] has been the subject of study in this laboratory.[5] The effect of the particle size to the attenuation of such waves in these gels is discussed upon the consideration of scattering and viscosity effects due to the spherical solids embedded in a liquid phase.[6]

The theoretical work originated in one of the publications is being expanded to eliminate a large number of complicated mathematical treatment to make it suited to experimental verifications.[7] The rather interesting phenomenon of variation in ultrasonic velocities with the frequency of the impressed waves, which is altogether absent in any solid or liquid in the strict sense, in the gelatinous state of matter can be explained on the consideration that particles of different sizes are present in a gel and that all the particles are, therefore, not equally effective in the propagation of the wave.[8]

In case larger particles are present they remain at rest when alternating air waves beat upon them whereas the smaller particles move to and fro with the impressed periodic force brought into play due to the passage of an ultrasonic wave. In that case the larger particles only contribute to the attenuation of the waves. These interesting cases are discussed in a forthcoming publication.

The belief among the colloid chemists that a highly viscous sol is essentially connected with a high degree of hydration of the colloid particles results in a conclusion that the colloidal particles ought to be more hydrated at higher temperatures. Ghosh and Banerji[9] observed that the sol of ferric phosphate at 50° C. becomes more quickly viscous when it is sewn with an already formed gel of ferric phosphate of the same concentration at the sol. The curves obtained by one of the authors (A.M.S.[10]) indicate that the changes in the state of gelation are not abrupt as evidenced in a few curves reproduced here (Figs. 1, 2, 3). This paper attempts to work out a theoretical curve along the lines of well-known treatment given by Pearson[11] and Atkin.[12] It is also intended to seek any correlations that might exist among different gels that have been studied.

2. Technique and Results

The ultrasonic pulse technique[13] was found suited to the investigation in gels because it was seen to have many advantages. The foremost being that an ultrasonic pulse is transmitted through the sample without doing any damage to it and it is comparatively easy in execution, calculation and more accurate. Also no extensive preparation of the samples is needed. The total reflection technique is fully described in previous publications.

Results obtained from the above applications are reproduced in Figs. 1, 2 and 3. Tables I-III are reproduced to show the values in the case of different gels of the Weimarn type that have been studied.

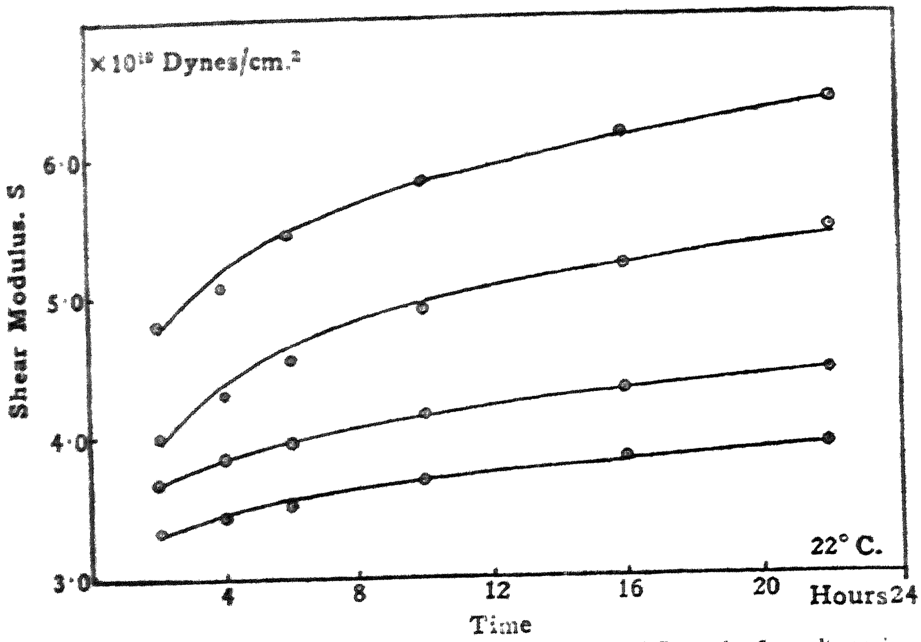

FIG. 1. Shows the variation of Shear modulus with time at 22° C. at the four ultrasonic frequencies, 2·50, 2·25, 1·25 and ·625 Mc/Sec.

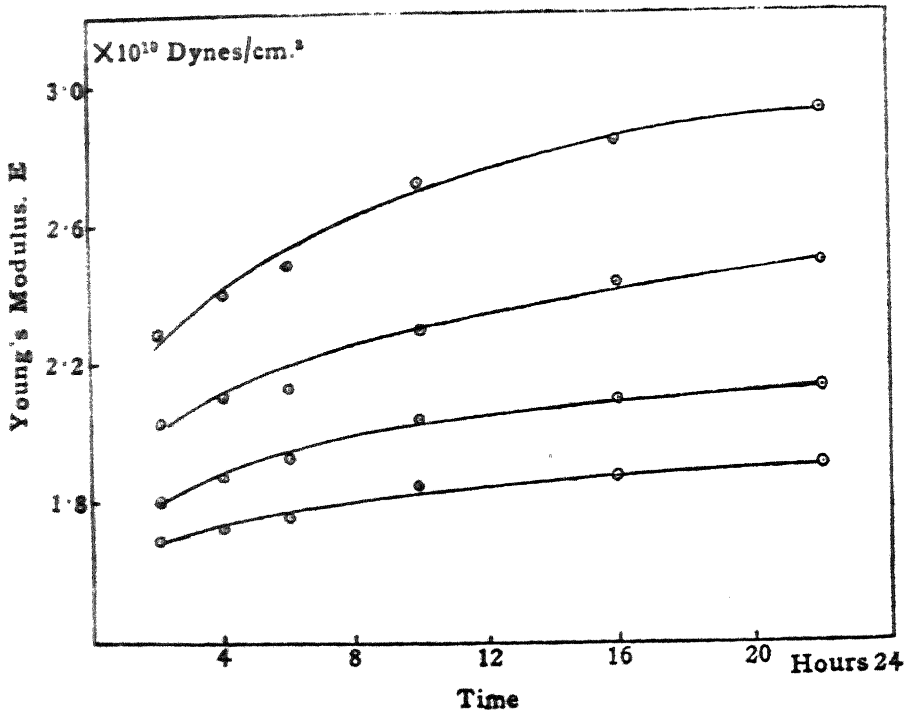

FIG. 2. Shows the variation of Young's modulus with time at 30° C. at the same four frequencies.

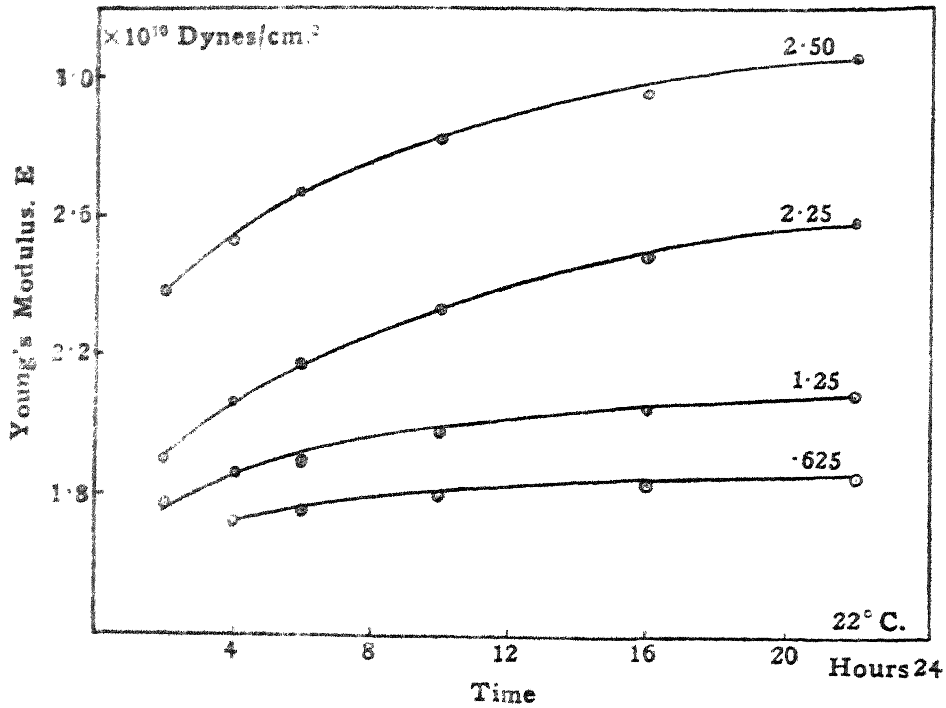

FIG. 3. Shows the variation of Young's modulus with time at 22° C.

TABLE I

Elasticities of Strontium Sulphate at Different Temperatures at 2·50 Mc/Sec.

No.	Temperature °C	V_e 10^5 cms./sec.	V_s	k	σ	E 10^{10} dynes/cm.^{2}	S
1	20	2.59	2·19	..	..	2·67	5·95
2	30	2·55	2·15	..	..	2·58	5·75
3	40	2·43	2·05	1·18	·78	2·46	5·48
4	50	2·29	1·93	..	..	2·07	4·63
5	60	2·19	1·85	..	..	1·90	4·23
6	70	2·14	1·78	1·20	·62	1·75	3·94

TABLE II

Strontium Phosphate at 2·25 Me/Sec.

No.	Temperature °C.	V_e 10^5 cms./sec.	V_s	k	σ	E 10^{10} dynes/cm.	S
1	22	2·60	2·09	1·22	·542	4·88	5·23
2	30	2·52	2·00	1·22	·54	4·68	4·12
3	45	2·40	1·83	1·23	·50	4·34	3·85
4	57	2·13	1·72	1·21	·43	4·06	3·53
5	70	2·01	1·65	1·22	·41	3·88	3·34

TABLE III

Elasticities of Barium Sulphate at 1·25 Me/Sec.

No.	Temperature °C.	V_e 10^5 cms./sec.	V_s	k	σ	E 10^{10} dynes/cm.	S
1	20	2·16	1·74	1·27	·32	4·94	3·61
2	30	2·15	1·72	1·26	·35	4·72	3·63
3	40	2·12	1·70	1·25	·39	4·39	3·58
4	50	2·08	1·68	1·24	·43	4·01	3·51
5	60	2·06	1·66	1·24	·43	3·90	3·46
6	70	2·06	1·66	1·24	·43	3·80	3·40

3. STATISTICAL DISCUSSION

In case we have n-pairs of values and it is required to represent them through a relationship of the type

$$y = a_0 + a_1x + a_2x^2 + \dots + a_px^p \tag{1}$$

the problem is to determine the constants $a_0, a_1, a_2 \dots a_p$, uniquely in terms of the known co-ordinates of the points, $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ so as to give the best possible fit after having determined the value of p according to the particular need. The best possible fit refers to the criteria

adopted, which, in our case would be the method of least squares incidentally the most universal in use. It means that a certain function U as defined below has a minimum value. They are

$$U = \Sigma (\xi_r)^2 = \Sigma\, [Y_r - a_0 - a_1X_r - \ldots . a_pX_r^p] \tag{2}$$

where

$$\xi_r = Y_r - y_r = Y_r - a_0 - a_1X_r - \ldots . - a_pX_r^p \tag{3}$$

defining the residual when we substitute the value of X_r in equation (1) to get the quantities

$$Y_r = a_0 + a_1X_r + a_2X_r^2 + \ldots . + a_pX_r^p \tag{4}$$

If $U = 0$, each of the residuals must be zero and the data are represented perfectly by equation (1). When this is not so, the value of $U > 0$ and the magnitude of U would represent the closeness of the fit. Then for the best fit the constants $a_0, a_1, \ldots, a_p$, must be so chosen as to give a least value to U. Let then this minimum value of U be U_0. In equation (2) if we put for the constants $a_0, a_1, \ldots, a_p$, the quantities $a_0 + \epsilon_0$, $a_1 + \epsilon_1$, $a_2 + \epsilon_2$, $\ldots, a_p + \epsilon_p$, we get U_1 which is always greater than U. for all values of $\epsilon_0, \epsilon_1, \ldots, \epsilon_p$.

Now

$$\begin{aligned} U_1 &= \Sigma\, [(Y - a_0 - a_1X - \ldots . - a_pX^p) - (\epsilon_0 + \epsilon_1X + \epsilon_2X^2 \\ &\quad + \ldots . + \epsilon_pX^p)]^2 \\ &= \Sigma\, [(Y - a_0 - a_1X - \ldots . - a_pX^p)^2 + (\epsilon_0 + \epsilon_1X + \epsilon_2X^2 \\ &\quad + \ldots . + \epsilon_pX^p)^2 \\ &\quad - 2\,(Y - a_0 - a_1X - \ldots . a_pX^p)\,(\epsilon_0 + \epsilon_1X + \ldots . + \epsilon_pX^p)] \end{aligned} \tag{6}$$

and if $U_1 \geqslant U_0$, we must have

$$\begin{aligned} &\Sigma\,(\epsilon_0 + \epsilon_1X + \ldots . + \epsilon_pX^p)^2 - 2\Sigma\, [(Y - a_0 - a_1X - \ldots . - a_pX^p) \\ &\qquad (\epsilon_0 + \epsilon_1X + \ldots . + \epsilon_pX^p)] \geqslant 0 \end{aligned} \tag{7}$$

for all values of $\epsilon_0, \epsilon_1, \ldots, \epsilon_p$. In case '$\epsilon$'s are small, the first term in (7) is negligible and the second must be zero or else a set of 'ϵ's can be selected to violate the condition laid down. Thus

$$(Y - a_0 - a_1X - \ldots . - a_pX^p)\ (\epsilon_0 + \epsilon_1X + \ldots . + \epsilon_pX^p) = 0 \tag{8}$$

Since (8) is true for all values of ϵ the coefficients of all the 'ϵ's must vanish and hence,

$$\Sigma(\mathrm{Y}) - a_0 n - a_1 \Sigma(\mathrm{X}) - a_2 \Sigma(\mathrm{X}^2) - \ldots - a_p \Sigma(\mathrm{X}^p) = 0$$
$$\Sigma(\mathrm{YX}) - a_0 \Sigma(\mathrm{X}) - a_1 \Sigma(\mathrm{X}^2) - a_2 \Sigma(\mathrm{X}^3) - \ldots - a_p \Sigma(\mathrm{X}^{p+1}) = 0$$
$$\Sigma(\mathrm{YX}^2) - a_0 \Sigma(\mathrm{X}^2) - a_1 \Sigma(\mathrm{X}^3) - a_2 \Sigma(\mathrm{X}^4) - \ldots - a_p \Sigma(\mathrm{X}^{p+2}) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
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$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Sigma(\mathrm{YX}^p) - a_0 \Sigma(\mathrm{X}^p) - a_1 \Sigma(\mathrm{X}^{p+1}) - a_2 \Sigma(\mathrm{X}^{p+2}) - \ldots - a_p \Sigma(\mathrm{X}^{2p}) = 0 \tag{9}$$

These are $(p+1)$ equations in the set (9) between $(p+1)$ quantities and hence they can be solved to give the constants $a_0, a_1, \ldots, a_p$ in terms of the known quantities $\Sigma(\mathrm{X}), \Sigma(\mathrm{X}^2) \ldots \Sigma(\mathrm{X}^{2p})$; $\Sigma(\mathrm{Y}), \Sigma(\mathrm{XY}) \ldots \Sigma(\mathrm{YX}^p)$.

If the rth moment is defined by

$$\mu_r' = \frac{1}{n} \Sigma(\mathrm{X}^r) \tag{10}$$

the set (9) can be expressed as,

$$\frac{1}{n}\Sigma(\mathrm{Y}) - a_0 - a_1\mu_1' - a_2\mu_2' - \ldots - a_p\mu_p' = 0$$
$$\frac{1}{n}\Sigma(\mathrm{YX}) - a_0\mu_1' - a_1\mu_2' - a_2\mu_3' - \ldots - a_p\mu'_{p+1} = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{n}\Sigma(\mathrm{YX}^p) - a_0\mu_p' - a_1\mu'_{p+1} - a_2\mu'_{p+2} - \ldots - a_p\mu_{2p}' = 0 \tag{11}$$

4. EXPONENTIAL CURVES

If the curve to be fitted is of the type

$$\mathrm{N} = a_0 e^{a_1 x} \tag{12}$$

then

$$\mathrm{Log}_e\, \mathrm{N} = \log_e a_0 + a_1 x \tag{13}$$

and putting

$$\mathrm{Log}_e\, \mathrm{N} = y,$$

we obtain,

$$y = \log_e a_0 + a_1 x \tag{14}$$

which is linear in x and y and in consequence there are only two constants to be determined, *i.e.*, $p = 2$ here.

5. YOUNG'S MODULUS AND TEMPERATURE

The following table illustrates the application of the previous theoretical discussion to the case of barium sulphate gel.

TABLE IV

Young's Modulus $\times 10^{10}$	$\text{Log}_e\,\eta = y$	T° C.	T^2	yT
5·22	·7177	22	484	15·79
4·76	·6776	30	900	20·32
4·41	·6444	45	2025	29·00
4·17	·6201	57	3249	35·35
4·08	·6107	70	4900	42·75

$\Sigma(y) = 3\cdot2705$ $\Sigma(T) = 224$ $\Sigma(T^2) = 11558$ $\Sigma(YT) = 143\cdot21$

Now

$$\Sigma\,(Y) - na_0 - a_1\Sigma\,(T) = 0$$

$$\Sigma\,(YT) - a_0\Sigma\,(T) - a_1\,\Sigma\,(T^2) = 0 \tag{15}$$

i.e.,

$$-224\,a_1 - 5a_0 + 3\cdot2705 = 0$$

$$-11558\,a_1 - 224a_0 + 143\cdot21 = 0 \tag{16}$$

Whence

$$a_1 = -\cdot0028$$

$$a_0 = \cdot779 \tag{17}$$

Now the equation is

$$y = \cdot779 - \cdot0028T$$

And in ordinary units

$$\text{Log}_e\,\eta = 10^{10\cdot779 - \cdot0028T}$$

Hence

$$\eta = 6\cdot01.\quad 10^{10}\,e^{-\cdot00667}$$

For *Strontium Sulphate*

TABLE V

Young's Modulus $\times 10^{10}$	$\text{Log}_e\ \eta = y$	T° C.	T^2	yT
2·67	·4265	20	400	8·530
2·58	·4116	30	900	12·348
2·46	·3909	40	1600	15·636
2·06	·3139	50	2500	15·695
1·90	·2788	60	3600	16·728
1·75	·2430	70	4900	17·010
	2·0647	270	13900	85·947

The equations to be solved in this case are

$$-270a_1 - 6a_0 + 2{\cdot}0647 = 0 \tag{18}$$

$$-13900a_1 - 270a_0 + 85{\cdot}947 = 0 \tag{19}$$

which give

$$a_1 = -\ {\cdot}0025 \tag{20}$$

$$a_0 = {\cdot}725,\ \ {\cdot}465 \tag{21}$$

Consequently the final value of η is expressed as

$$\eta = 7{\cdot}70.\quad 10^{10}\, e^{-{\cdot}00574T}$$

For *Strontium Phosphate*

TABLE VI

Young's Modulus $\times 10^{10}$	$\text{Log}_e\ \eta = y$	T° C.	T^2	yT
4·88	·6884	22	484	15·15
4·68	·6702	30	900	20·10
4·34	·6375	45	2025	28·74
4·06	·6085	57	3249	34·68
3·88	·5888	70	4900	41·12
	3·1934	224	11558	139·79

The equations are,

$$-224\,a_1 - 5a_0 + 3\cdot 19 = 0 \qquad (22)$$

$$-11558a_1 - 224a_0 + 139\cdot 8 = 0 \qquad (23)$$

Consequently

$$a_1 = -\cdot 0022 \qquad (24)$$

$$a_0 = \cdot 791 \qquad (25)$$

For *Thorium Phosphate*

TABLE VII

Young's Modulus $\times 10^{10}$	$\text{Log}_e\ \eta = y$	T° C.	T^2	yT
7·50	·8751	20	400	17·50
6·82	·8338	35	1225	29·28
5·57	·7459	50	2500	37·30
5·07	·7050	64	4096	45·10
4·83	·6839	74	5476	50·62
	4·0437	243	13697	179·80

From the above tables the following equations result

$$-243a_1 - 5a_0 + 4\cdot 044 = 0 \qquad (26)$$

$$-13697a_1 - 243a_0 + 179\cdot 80 = 0 \qquad (27)$$

which give

$$a_1 = -\cdot 0085\ a_0 = 1\cdot 30 \qquad (28)$$

For *Iron Silicate* (No. 1)

This is a gel of a different group and it may be of some importance to compare it with those of the Weimarn type seen above.

TABLE VIII

Young's Modulus $\times 10^{10}$	$\text{Log}_e\ \eta = y$	T° C.	T^2	yT
2·23	·3483	22	484	7·66
1·65	·2175	30	900	6·53
1·14	·0569	45	2025	2·57
0·81	·0338	57	3249	1·93
0·56	·0236	70	4900	1·65
	·6801	224	11558	20·34

The equations follow

$$-224a_1 - 5a_0 + \cdot 6801 = 0 \qquad (29)$$

$$-11558a_1 - 224a_0 + 20\cdot 34 = 0 \qquad (30)$$

and

$$a_0 = \cdot 45, \qquad a_1 = - \cdot 0061 \qquad (31)$$

For *Iron Silicate* (No. 2)

TABLE IX

Young's Modulus $\times 10^{10}$	$\text{Log}_e \eta = y$	T° C.	T²	yT
4·25	·6284	20	400	12·57
3·60	·5563	30	900	16·69
3·12	·4942	45	2025	22·24
2·78	·4440	57	3242	25·31
2·59	·4133	70	4900	28·93
	2·5362	222	11558	105·74

The resulting equations are,

$$-222a_1 - 5a_0 + 2\cdot 536 = 0 \qquad (32)$$

$$-11558a_1 - 222a_0 + 105\cdot 74 = 0 \qquad (33)$$

Which yield,

$$a_1 = - \cdot 0041, \qquad a_0 = \cdot 69$$

[In column 2 of Tables IV–IX above, logarithms are taken to the base ten in place of Naperian. Consequently, as a little mathematics shows, the values of a_0 and a_1 will come out to be those obtained divided by Log_e 10 *i.e.*, 2·30.]

6. DISCUSSION OF RESULTS

The statistical correlation of gel data, considering the variation of Young's modulus with temperature, has been carried out in Tables IV-IX. As a result of Equation 12 the constants a_0 and a_1 are calculated from experimental data. Table X shows at a glance the different results obtained.

The remarkable agreement seen in Table X to the effect that the values for six Weimarn gels do not show a variation greater than 12 per cent. among $BaSO_4$, $SrSO_4$ and $SrPO_4$ gives reason to conclude of some common

TABLE X

No.	Gel	a_0	a_1	Reference
1	Barium Sulphate ..	·779	—·0028	Weimarn gels A. Data given in Tables IV, V, VI.
2	Strontium Sulphate ..	·725	—·0025	
3	Strontium Phosphate ..	·791	—·0022	
4	Thorium Phosphate ..	1·300	—·0085	Table VII
5	Iron Silicate I	·45	—·0061	Table VIII
6	Iron Silicate II	·69	—·0047	Table IX
7	Barium Carbonate ..	·756	—·0031	Weimarn gels B. Data not given here.
8	Calcium Sulphate ..	·732	—·0026	
9	Magnesium Phosphate ..	·716	—·0024	

ground in their behaviour to elastic deformation forces. Similarly gels in the other group B taken from our calculations not shown here, do not give a variation of more than 11 per cent. in the values of *as*.

In a similar manner the values of a_0 are uniform for all Weimarn gels.

But this agreement in the gels of same group does not extend to others like iron silicate or thorium phosphate. In the former case a_1 assumes a value 2 to 3 times greater than in the Weimarn group. Similar disparity is observed in a_0 too, though they do not show such a high deviation. This implies that the zero state value for all gels are not so very different but their behaviour in transformation due to temperature variation undergoes different paths. Gels of the same group however traverse almost the same path and attain different values of E simultaneously (with respect to T° C.).

The fundamental one-ness of structure in gels cannot, therefore, be percieved during the transformation temperature states where different groups will be found to have totally diverse values. To seek an identity in the structure of gels one shall have to probe deeper than their studies at normal room temperatures.

Our studies here, though of a very limited nature, purport to establish that it may be possible to show this identity at zero temperature states where a_0's might have a more uniform value.

This paper is the first step in this direction and as such we are not in a hurry either to positively assert to have achieved an identity in gel structures or categorically deny all such attempts to be misconceived. In view of these limited results we are in favour of the former in case a zero state of temperature is considered and to the latter at any arbitrary temperature.

Further work is in progress where the problem is dealt with from the standpoint of another important parameter, *viz.*, the time of setting or hardening of the gels, which is likely to throw more light on the problem.

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GENERALIZATION OF A THEOREM OF NEWSOM

By Nirmala Pandey

(Department of Mathematics, Allahabad University)

Newsom[1] has proved a theorem concerning the behaviour of the series

$$\sum_{m=0}^{\infty} g(m)\, z^m, \text{ radius of convergence} = \infty,$$

when the absolute value of z becomes large. The coefficient $g(m)$ occurring in the general term of the power series may be regarded as a function $g(\omega)$ of the complex variable $\omega = (x + iy)$ and as such satisfies the following two conditions:

(*a*) it is single valued and analytic throughout the finite ω-plane;

(*b*) it is such that for all values of x and y one may write

$$| g(x + iy) | < K e^{k\pi |y|}$$

where K is a constant independent of x and y, and k is any given positive integer.

In the present paper, I wish to prove a similar theorem under less restricted conditions on the coefficient $g(m)$.

Theorem.—Let it be assumed that the coefficient $g(m)$ occurring in the general term of the power series

$$\Omega(z) = \sum_{m=0}^{\infty} e^{Ai(k+1)m}\, g(m)\, z^m, \text{ radius of convergence} = \infty, \tag{1}$$

when considered as a function $g(\omega)$ of $\omega\,(= x + iy)$ satisfies the following two conditions:

(*a*) it is single valued and analytic throughout the finite ω-plane;

(*b*) it is such that for all values of x and y, one may write

$$| g(x + iy) | < K_\epsilon \exp\{(\overline{k+1}\,A - \pi - \epsilon)\, | y | \} \tag{2}$$

and

$$| g(x - iy) | < K_\epsilon' \exp\{(2\pi\overline{k+1} - A\overline{k+1} - \pi - \epsilon)\ | y | \}$$

(K_ϵ and K_ϵ' being constants).

Then the function $\Omega(z)$ defined by the series (1) when considered for all values of z satisfying the condition $-\pi < \arg z < \pi$ may be expressed in the form

$$\Omega(z) = \int_{-l-\frac{1}{2}}^{\infty} e^{Ai\overline{k+1}x} g(x) z^{x} \frac{(1-\cos 2\pi kx + i \sin 2\pi kx)}{(\cos 2\pi x - 1 + i \sin 2\pi x)} dx \qquad (3)$$

$$-\sum_{n=-l}^{-1} e^{Ai\overline{k+1}n} g(n) z^{n} + \xi_k(l, z),$$

where l is any positive integer chosen arbitrarily and the expression $\xi_k(l, z)$ is such that

$$\sin_{|z|\to\infty} z^{l} \xi_k(l, z) = 0 \qquad (4)$$

irrespective of the value chosen for l.

Proof.—Considering the integral

$$\int_{C_m} \frac{e^{Ai\overline{k+1}\omega} g(\omega) z^{\omega}}{(e^{2\pi i\omega} - 1)^{k+1}} d\omega \qquad (5)$$

where the path of integration C_m is any closed (finite) contour enclosing the points $\omega = -l, -l+1, -l+2, \ldots, -1, 0, 2, \ldots -n$, and applying Cauchy's integral theorem we get

$$\sum_{m=0}^{m} e^{Ai\overline{k+1}m} g(m) z^{m}$$

$$= (2\pi i)^{k} k! \sum_{j=1}^{k} b_j \int_{C_m} \frac{e^{2\pi i r_j \omega} \mathrm{I}(\omega, z)}{(e^{2\pi i\omega} - 1)^{k+1}} d\omega$$

$$-\sum_{n=-l}^{-1} e^{Ai\overline{k+1}n} g(n) z^{n}, \qquad (6)$$

where

$$\mathrm{I}_j(\omega, z) = \int_0^{\omega} \exp(-r_j 2\pi i\omega + Ai\overline{k+1}\omega) g(\omega) z^{\omega} d\omega, \qquad (7)$$

and

$$r_j = k + 1 - j, \quad 1 \leqslant j \leqslant k. \qquad (8)$$

Also b_j is given by

$$b_j = (-1)^{k-j} \frac{1}{(2\pi i)^{k-1}} \frac{1}{(j-1)!(k-j)!}. \qquad (9)$$

In order to study the right-hand member of (6) we first study k integrals of the form

$$\int_{C_m} \frac{e^{2\pi i r_j \omega} \int_0^{\omega} e^{-r_j 2\pi i \omega + Aik + i\omega}\, g(\omega)\, z^{\omega}}{\{\exp(2\pi i \omega) - 1\}^{k+1}}\, d\omega. \tag{10}$$

If we put $z = \rho(\cos\phi - i\sin\phi)$, $I_j(\omega, z)$ can be written in the form

$$I_j(\omega_1, z) = \int_0^{\omega} [\exp\{(2r_j\pi - \phi - Ak - 1)y + i(y\log\rho + \phi - 2r_j\pi x$$

$$+ Ak + 1x)\}\, \rho^x g(\omega)]\, d\omega \tag{11}$$

We may take the path of integration in (11) to be that which starts from the origin and goes to the point $\omega = x$ and then proceeds parallel to the pure imaginary axis to the point $\omega = x + iy$. $I_j(\omega, z)$ can then be written in the form

$$I_j(\omega, z) = R_j(\rho, \phi, x) + S_j(\rho, \phi, x, y), \tag{12}$$

where

$$R_j(\rho, \phi, x) = \int_0^x \rho^x e^{(\phi - 2r_j\pi + Ak + 1)xi}\, g(x)\, dx \tag{13}$$

and

$$S_j(\rho, \phi, x, y) = i\rho^x \int_0^y [g(x + iy)\, e^{(2\pi r_j - \phi - Ak + 1)y}$$

$$\times\; e^{(y\log\rho + \phi - 2\pi r_j x + Ak + 1\, x)i}]\, dy. \tag{14}$$

Having confined z to any finite region B of the z-plane in which $-\pi + \epsilon < \phi < \pi - \epsilon$, ϵ being positive and arbitrarily small and ρ being bounded, if we confine x and y to any strip of finite width (so that x is bounded) drawn in the ω-plane parallel to the y-axis, we may write according as $y > 0$ or $y < 0$,

$$|S_j(\rho, \phi, x, y)| < M_1\, e^{(2r_j\pi - \epsilon)y}, \quad y > 0, \tag{15}$$

and

$$|S_j(\rho, \phi, x, y)| < M_2\, e^{(2\,\overline{k+1}\,\pi - 2r_j\pi - \epsilon)\,|y|}, \quad y < 0,$$

where M_1 and M_2 are positive constants independent of ρ, ϕ, x and y.

We may take C_m to be the rectangle formed by the lines $\omega = -l - \frac{1}{2} + iy$, $\omega = n + \frac{1}{2} + iy$, $w = x + ip$ and $\omega = x + iq$, where we take l to be an arbitrarily large positive even integer and p and q to be arbitrarily large positive and negative quantities.

Denoting by A_j the contribution to (10) from the side upon which $\omega = x + ip$ we have

$$A_j = \int_{n+1/2}^{-l-1/2} \frac{\exp\{2r_j\pi i(x+ip)\}\,[R_j(\rho,\phi,x) + S_j(\rho,\phi,x,p)]}{\{\exp(2\pi i\,\overline{x+ip}) - 1\}^{k+1}}\,dx \tag{16}$$

and taking the modulus on both the sides we may write

$$|A_j| \leq \left| \int_{n+1/2}^{-l-1/2} \frac{R_j(\rho,\phi,x)}{e^{2r_j\pi p}}\,dx \right| + \left| \int_{n+1/2}^{-l-1/2} \frac{S_j(\rho,\phi,x,p)}{e^{2r_j\pi p}}\,dx \right|. \tag{17}$$

Since $R_j(\rho,\phi,x)$ is bounded for all values of ρ, ϕ and x with which we are concerned in (17), the first integral on the right of (17) approaches zero as p approaches infinity.

Also the absolute value of the integrand in the second integral on the right of (17) cannot become infinite to an order higher than that of $e^{-\epsilon p}$ for large and positive values of p. We therefore have

$$\lim_{p\to\infty} A_j = 0.$$

Similarly, it can be shown that

$$\lim_{p\to\infty} B_j = 0,$$

where B_j is the contribution to the integral (10) arising from that side of C_m upon which $\omega = x + iq$, $(q < 0)$.

If C_j be the contribution arising from the side $\omega = n + \frac{1}{2} + iy$, we may write

$$C_j = (-1)^{r_j}\left\{ i\,R_j(\rho,\phi,n+\tfrac{1}{2}) \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)}{(-e^{-2\pi y} - 1)^{k+1}}\,dy \right\}$$

$$-(-1)^{r_j} \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)\,S_j(n+\frac{1}{2},y,z)}{(-e^{-2\pi y} - 1)^{k+1}}\,dy. \tag{18}$$

The integral $\displaystyle\int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)}{(-e^{-2\pi y} - 1)^{k+1}}\,dy$ which we may denote by H_j, can be evaluated directly.

We may write

$$H_j = \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)}{(-e^{-2\pi y}-1)^{k+1}}\, dy = (-1)^{k+1} \frac{1}{2\pi} \frac{(k-j)!\,(j-1)!}{k\,!}. \quad (19)$$

Also the absolute value of the integrand in t e second integral on the right of (18) is less than

$$\frac{M_1'\, \rho^{n+\frac{1}{2}}\, e^{-\epsilon y}}{(e^{-2\pi y}-1)^{k+1}} \text{ or } \frac{M_2'\, \rho^{n+\frac{1}{2}}\, e^{(2\overline{k+1}\pi-\epsilon)|y|}}{(e^{2\pi|y|}-1)^{k+1}}, \quad (20)$$

where M_1' and M_2' are assignable positive constants and the two expressions in (20) are to be taken respectively according as $y > 0$ or $y < 0$. Now since $(e^{-2\pi y}-1)^{-k-1}$ and $(e^{2\pi|y|}-1)^{-k-1}$ remain bounded according as $y > 0$ or $y < 0$ respectively, the integral in question converges uniformly for all values of z lying in the region B of the z-plane. Moreover, if ρ be restricted only to those values of B which lie inside the unit circle about the origin, the limit of the integral in question approaches zero as n approaches infinity. We may therefore write

$$C_j = i(-1)^{r_j} H_j R_j(n+\tfrac{1}{2}, \rho, \phi) + \eta_j(n, z) \quad (21)$$

where for these values of z in B for which $|z| < 1$ we can write

$$\lim_{n\to\infty} \eta_j(n, z) = 0.$$

Lastly we consider the contribution D_j to (10) arising from integrating over the side $-l-\frac{1}{2}+iy$ of the rectangle C_m. We have

$$D_j = (-1)^{r_j} i H_j R_j(\rho, \phi, -l-\tfrac{1}{2}) + y_j(l, z), \quad (22)$$

where $y_j(l, z)$ is given by

$$y_j(l, z) = (-1)^{r_j+1} \int_{-\infty}^{\infty} \frac{e^{-2r_j\pi y}\, S_j(\rho, \phi, -l-\frac{1}{2}, y)}{(-e^{-2\pi y}-1)^{k+1}}\, dy. \quad (23)$$

It is evident that an analysis similar to that applied to the second integral of (18) can also be applied to the integral in (23). In fact, the integrand of this integral is less in absolute value than

$$\frac{M_2\, \rho^{-l-\frac{1}{2}}\, e^{-\epsilon y}}{(e^{-2\pi y}+1)^{k+1}} \quad \text{or} \quad \frac{M_2\, \rho^{-l-\frac{1}{2}}\, e^{(2\pi\overline{k+1}-\epsilon)|y|}}{(e^{2x|y|}+1)^{k+1}} \quad (24)$$

according as $y > 0$ or < 0. Hence it follows that $y_j(l, Z)$ is uniformly convergent throughout the portion of B in which $|z| = \rho \geq \rho_1 > 0$ where

ρ_1 is arbitrarily small and is such that for these values of z in the portion of B just specified we shall have

$$\lim_{|z|\to\infty} z^l y_j(l, z) = 0 \tag{25}$$

Also we may write

$$R_j(\rho, \phi, n+\tfrac{1}{2}) - R_j(\rho, \phi, -l-\tfrac{1}{2})$$

$$= \int_{-l-1/2}^{n+1/2} g(x)\, z^x \exp\{-r_j 2\pi x i + Ai\overline{k+1}x\}\, dx \tag{26}$$

Adding now A_j, B_j, C_j and D_j together, we have

$$\int_{C_m} \frac{e^{r_j 2\pi i\omega} \int_0^{\omega} e^{-r_j 2\pi i\omega}\, g(\omega)\, z^{\omega}\, e^{Ai\omega(k+1)}\, d\omega}{(e^{-2\pi y}+1)^{k+1}}\, d\omega$$

$$= i(-1)^{r_j} H_j \int_{-l-1/2}^{n+1/2} g(x)\, z^x \exp\{-2r_j\cdot\pi x i + Ai\overline{k+1}x\}\, dx$$

$$+ \eta_j(n, z) + y_j(l, z) \tag{27}$$

where $\eta_j(n, z)$ and $y_j(l, z)$ are defined by (21) and (25).

Equation (6) can now be written in the form

$$\sum_{m=0}^{\infty} e^{Ai\overline{k+1}m}\, g(m)\, z^m$$

$$= \sum_{j=1}^{k} i(2\pi i)^k k!(-1)^{r_j} l_j H_j \int_{-l-1/2}^{n+1/2} g(x)\, z^x \exp(-r_j\cdot 2\pi x i + Ai\overline{k+1}x)\, dx$$

$$+ \sum_{j=1}^{k} (2\pi i)^k k!\, b_j \eta_j(n, z) + \sum_{j=1}^{k} (2\pi i)^k k!\, b_j y_j(l, z)$$

$$- \sum_{n=-l}^{-1} e^{Ai\overline{k+1}n}\, g(n)\, z^n. \tag{28}$$

Recalling the values of b_j and H_j given by (9) and (19) respectively and making use of the equality

$$\sum_{j=1}^{k} e^{-2r_j\pi x i} = \frac{1-\cos 2\pi kx + i\sin 2\pi kx}{\cos 2\pi x - 1 + i\sin 2\pi x}, \tag{29}$$

We may write

$$\sum_{m=0}^{\infty} e^{Ai\,\overline{k+1}\,m}\; g(m)\, z^m$$

$$= \int_{-l-1/2}^{n+1/2} (-1)^{k-1} \left\{ g(x)\, z^x\; e^{Ai\,\overline{k+1}\,x}\, \frac{1-\cos 2\pi kx - i\sin 2\pi kx}{\cos 2\pi x + i \sin 2\pi x - 1}\, dx \right\}$$

$$- \sum_{n=-l}^{-1} e^{Ai\,\overline{k+1}\,n}\, g(n)\, z^n + \eta(n,z) + \xi_k(l, z), \tag{30}$$

where for all z in B having $|z| < 1$, we can write

$$\operatorname*{Lim}_{n\to\infty} \eta(n,z) = 0$$

and for all z in B for which $|z| \geqslant \rho_1 > 0$, $\xi_k(l, z)$ is uniformly convergent and we have

$$\operatorname*{Lim}_{|z|\to\infty} z^l\, \xi_k(l, z) = 0$$

If therefore we now confine z to values in B for which $0 < \rho_1 \leq z < 1$ and then allow n to approach infinity we can write

$$\Omega(z) = \sum_{m=0}^{\infty} e^{Ai\,\overline{k+1}\,m}\; g(m)\, z^m$$

$$= \int_{-l-1/2}^{\infty} e^{Ai\,\overline{k+1}\,x}\; g(x)\, z^x\, \frac{\{1-\exp(-2\pi i\, kx)\}}{\{\exp(2\pi xi - 1\}}\, dx$$

$$- \sum_{n=-l}^{-1} g(n)\, z^n\; e^{Ai\,\overline{k+1}\,n} + \xi_k(l, z). \tag{31}$$

Since by hypothesis the radius of convergence of the series is infinite, the left-hand member of (31) in an analytic function of z for all finite z. Also the right-hand member is an analytic function of z for all z in B for which $|z| \geqslant \rho_1 > 0$ as follows from the uniform convergence of ξ_k (l, z) for all such z. By principles of analytic continuation it follows therefore that (31) holds true for all z in B for which $|z| \geqslant \rho_1 > 0$ and the theorem is established.

I wish to thank Prof. P. L. Srivastava for his guidance and help.

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ON THE SINGULARITIES OF A CLASS OF LAPLACE-ABEL INTEGRAL

By Nirmala Pandey

(Department of Mathematics, Allahabad University)

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1. The object of this paper is to study the singularities of the function represented by

$$J_3(S) = \int_0^\infty \phi_1(z)\,\phi_2(z)\,e^{-Sz}\,dz$$

in terms of the singularities of

$$J_1(S) = \int_0^\infty \phi_1(z)\,e^{-Sz}\,dz$$

and

$$J_2(S) = \int_0^\infty \phi_2(z)\,e^{-Sz}\,dz$$

where $\phi_1(z)$ and $\phi_2(z)$ are analytic functions of $z\,(= re^{i\theta})$ in the angular region $|\theta| \leqslant \alpha$, $\alpha > 0$, and each one of them is of order e^{kr}, uniformly throughout this angular region.

2. To do this we shall first prove the following theorem:

Theorem 1.—Suppose that

$$h_1(S) = \sum_{n=0}^{\infty} a_n\,e^{-Sn}, \qquad \sigma > K_1,\ |t| \leqslant \Pi;$$

and

$$h_2(S) = \sum_{n=0}^{\infty} b_n\,e^{-Sn}, \qquad \sigma > K_2,\ |t| \leqslant \Pi;$$

and that the singularities of $h_1(S)$ and $h_2(S)$ are known. What can be said about the singularities of the function

$$H(S) = \Sigma\, a_n\, b_n\, e^{-Sn}$$

the coefficients of which are the products of those in the given series?

The general result is that the only possible singularities of H (S) in the strip $|t| \leqslant \Pi$ are those obtained by adding the singularities of $h_1(S)$ to those of $h_2(S)$.

This result is obtained by transforming Hadamard's multiplication theorem for power series $\Sigma a_n z^n$, $\Sigma b_n z^n$ and $\Sigma a_n b_n z^n$ into the s-plane by the substitution $z = e^{-s}$. Also it is understood that whether the sum of the imaginary parts of these singularities is greater than or less than Π, the corresponding point, lying inside the strip $|t| \leqslant \Pi$, has to be taken.

Having thus established a relation between the singularities of h_1 (S), h_2 (S) and H (S), we now prove our main theorem.

Theorem 2.—If $\phi_1(z)$ and $\phi_2(z)$ are analytic functions of $z\,(= re^{i\theta})$ in the region $|\theta| \leqslant a$, $a \geqslant \frac{\Pi}{2}$, and $\phi_1(z) = 0\,(e^{k_1 r})$ and $\phi_2(z) = 0\,(e^{k_2 r})$, K_1 and K_2 being positive throughout this region, then J_3 (S) defined by the integral $\int_0^\infty \phi_1(z)\,\phi_2(z)\,e^{-Sz}\,dz$ has no other singularities than possibly those obtained by adding the singularities of

$$J_1(S) = \int_0^\infty \phi_1(z)\,e^{-Sz}\,dz$$

to those of

$$J_2(S) = \int_0^\infty \phi_2(z)\,e^{-Sz}\,dz.$$

Let us first study the functions J_1 (S), J_2 (S) and J_3 (S) in the s'-plane where $s' = Ks$, K being a positive number such that KK_1 and KK_2 are each less than $\frac{\Pi}{2}$. This only means that instead of studying the singularities of J_1 (S) in the s-plane, we study the singularities of $J_1\left(\frac{s'}{K}\right)$ in the s'-plane, where

$$J_1(S) = J_1\left(\frac{s'}{K}\right) = \int_0^\infty \phi_1(z)\; e^{-\frac{s'}{K}z}\; dz$$

$$= \int_0^\infty \phi_1(K\xi)\,e^{-s'\xi}\,K\,d\xi,$$

$$= K\int_0^\infty f_1(\xi)\,e^{-s'\xi}\; d\xi,$$

where

$$f_1(\xi) = \phi_1(K\xi)$$

and

$$f_1(\xi) = O\,(e^{K_1|K\xi|}) = 0\;(e^{K_1K|\xi|}),\quad KK_1 < \frac{\Pi}{2}.$$

This shows that our theorem is true for all K_1 and K_2, if it is true when each one of them is less than $\frac{\Pi}{2}$, so that

$$\phi_1(z)\,\phi_2(z) = 0\ \{e^{(K_1+K_2)r}\},$$

$(K_1 + K_2)$ being less than Π.

To prove the theorem when K_1 and K_2 are less than $\frac{\Pi}{2}$, consider the series,

$$F_1(S) = \Sigma\,\phi_1(n)\,e^{-Sn}$$

$$F_2(S) = \Sigma\,\phi_2(n)\,e^{-Sn} \text{ and}$$

$$F_3(S) = \Sigma\,\phi_1(n)\,\phi_2(n)\,e^{-Sn}.$$

By a theorem of Prof. Srivastava,[1] if $\phi(z)$ is an analytic function of z in the angular region $|\theta| \leqslant \alpha$, $\alpha \geqslant \frac{\Pi}{2}$, and $\phi(z) = 0\,(e^{Mr})$, M being less than Π, throughout this angular region, then we may write

$$F(S) = \Sigma\,\phi(n)\ e^{-Sn} = \int^{\infty(-\pi/2)} \frac{\phi(z)\,e^{-Sz}}{(e^{2\pi iz}-1)}\,dz - \int^{\infty(\pi/2)} \frac{\phi(z)\,e^{-Sz}}{(e^{2\pi iz}-1)}\,dz \qquad (2.1)$$

$$= \int^{\infty} \phi(x)\,e^{-Sx}\,dx + \int^{\infty(\pi/2)} \frac{\phi(z)\,e^{-Sz}}{e^{-2\pi iz}-1}\,dz + \int^{\infty(-\pi/2)} \frac{\phi(z)\,e^{-Sz}}{e^{2\pi iz}-1}\,dz$$

the right side giving the analytic continuation of the function F (S) represented by Dirichlet's series in the region $\sigma \geqslant M + \delta > M$, $|t| < \Pi$.

The integral

$$\int^{\infty(-\pi/2)} \frac{\phi(z)\,e^{-Sz}}{(e^{2\pi iz}-1)}\,dz$$

is absolutely and uniformly convergent over the region

$$t \geqslant -(2\Pi - M) + \delta > -(2\Pi - M), \quad \sigma \text{ bounded}.$$

Similarly the integral

$$\int^{\infty(\pi/2)} \frac{\phi(z)\,e^{-Sz}}{(e^{-2\pi iz}-1)}\,dz$$

is absolutely and uniformly convergent over the region

$$(2\Pi - M) \geqslant t + \delta > t, \qquad \sigma \text{ bounded}.$$

Hence both these integrals represent an analytic function of S in any finite part of the strip $|t| \leqslant \Pi$. Let us denote their sum by $g(S)$. Equation (2.1) then becomes

$$F(S) = g(S) + \int^{\infty} \phi(z)\, e^{-Sz}\, dz$$

where $g(S)$ is an analytic function of S in the strip $|t| \leqslant \Pi$

Equation (2.1) has been obtained on the assumption that S is real, positive and greater than M. But since

$$F(S) - \int^{\infty} \phi(z)\, e^{-Sz}\, dz = g(S)$$

is true for all values of S in the strip $|t| \leqslant \Pi$, we may, by the principle of analytic continuation, say that the finite singularities of $F(S)$ are identical to those of the integral $\int^{\infty} \phi(z)\, e^{-Sz}\, dz$ lying in the strip $|t| \leqslant \Pi$.

Hence it follows that the functions $F_1(S)$, $F_2(S)$ and $F_3(S)$ have the same singularities in the strip $|t| \leqslant \Pi$ as $J_1(S)$, $J_2(S)$ and $J_3(S)$ respectively. Also by theorem (1), $F_3(S)$ has no other singularities in the strip $|t| \leqslant \Pi$, then possibly those obtained by adding the singularities of $F_1(S)$ to those of $F_2(S)$. The same result therefore holds among the singularities of $J_1(S)$, $J_2(S)$ and $J_3(S)$. Our main theorem is thus established.

This result is a generalization of Hurwitz theorem[2] obtained when $\phi_1(z)$ and $\phi_2(z)$ are integral functions of z. Also we get another proof of that theorem. For if

$$
\begin{aligned}
J_1(S) &= \int_0^{\infty} \phi_1(z)\, e^{-Sz}\, dz \\
&= \int_0^{\infty} \sum \frac{a_n z^n}{n!}\, e^{-Sz}\, dz \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} z^n\, e^{-Sz}\, dz \\
&= \sum_{n=0}^{\infty} \frac{a_n}{S^{n+1}} = g_1(S) \text{ sa}
\end{aligned}
$$

$$J_2(S) = \sum_{n=0}^{\infty} \frac{b_n}{S^{n+1}} = g_2(S) \text{ say:}$$

and

$$J_3(S) = \sum (a_0 b_n + c_n' a_1 b_{n-1} + \ldots + a_n b_0) \frac{1}{S^{n+1}},$$

$$= g_3(S) \text{ say}$$

then whatever relation is true among the singularities of $J_1(S)$, $J_2(S)$ and $J_3(S)$ is also true among the singularities of $g_1(S)$, $g_2(S)$ and $g_3(S)$.

Theorem 3.—If

$\lambda(\omega)$ be a branch of an analytic function of ω ($= x + iy = \beta + \rho e^{i\phi}$, $p - 1 < \beta < p$) in the angle $|\phi| \leqslant \alpha$, $\alpha \geqslant \frac{\Pi}{2}$ and $\lambda(\omega) = 0(\rho)$ uniformly in this angle as $\rho \to \infty$;

$\lambda(x)$ be an L-function such that it is positive for $x \geqslant p$ and steadily tends to infinity with x;

$\lambda'(\omega) = 0(1)$ as $\rho \to \infty$ uniformly in the angle $|\phi| \leqslant \alpha$;

$\phi(z)$ be an analytic function of $z (= re^{i\theta})$ in the angle $|\theta| \leqslant \alpha$, $\alpha \geqslant \frac{\Pi}{2}$ and satisfy the relation $\phi(z) = 0(e^{Mr})$, throughout this angle;

$\phi[\lambda(\omega)]$ be an analytic function of ω in the angle $|\phi| \leqslant \alpha$, and $|[\lambda(\omega)\phi\{\lambda(\omega)\} e^{-k\lambda(\omega)} (\omega - \beta)]| \to 0$ as $\rho \to \infty$ uniformly on the arc $|\omega - \beta| = r$, $|\phi| \leqslant \alpha$ for some positive values of k; then if

$$\lambda(\theta) \equiv \lim_{r \to \infty} \frac{\log |f(r e^{i\theta})|}{r}, \ (|\theta| \leqslant \alpha),$$

be finite, the only possible singularities of the series

$$H_3(S) = \Sigma \phi_1(\lambda_n) \phi_2(\lambda_n) \lambda'_n e^{-S\lambda_n}, \tag{2.2}$$

are those obtained by adding the singularities of

$$H_1(S) = \Sigma \phi_1(\lambda_n) \lambda'_n e^{-S\lambda_n} \tag{2.3}$$

to these of

$$H_2(S) = \Sigma \phi_2(\lambda_n) \lambda'_n e^{-S\lambda_n}, \tag{2.4}$$

where $\phi_1(z)$ and $\phi_2(z)$ are functions similar to $\phi(z)$ and satisfy all the conditions of the theorem.

By a theorem of Prof. Srivastava,[3] all the finite singularities of $H_1(S)$, $H_2(S)$ and $H_3(S)$ are identical to the singularities of $J_1(S)$, $J_2(S)$ and $J_3(S)$, respectively so that the theorem is at once established.

3. We have not been able to prove any result similar to that established in Theorem 2, if $\phi_1(z)$ and $\phi_2(z)$ are functions analytic in regions less than half-planes. What we have proved is the following theorem:

Theorem 4.—If $\phi_1(z)$ and $\phi_2(z)$ are analytic in the angular region $|\theta| \leqslant \alpha < \frac{\Pi}{2}$, and satisfy the following conditions:

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{\log|\phi_1(z)|}{r} = \lambda_1(\theta); \qquad |\theta| \leqslant \alpha < \frac{\Pi}{2},$$

$$|\phi_2(z)| < e^{\{\lambda_2(\theta)+\epsilon\}r} \qquad \text{for all sufficiently large } r,$$

and

$$|\phi_2(z)| > e^{\{\lambda_2(\theta)-\epsilon\}r} \qquad \text{for all large } r,$$

$\lambda_1(\theta)$, $\lambda_2(\theta)$ being finite for $|\theta| \leqslant \alpha$, then $J_3(S) = \int_0^\infty \phi_1(z)\phi_2(z)e^{-Sz}\,dz$ is an analytic function of S in the region lying exterior to the curve Σ which is the envelope of the family of lines

$$\sigma \cos\theta - t\sin\theta = \lambda_1(\theta) + \lambda_2(\theta), \qquad |\theta| \leqslant \alpha$$

and such points on Σ as correspond to continuities of $\{\lambda_1(\theta) + \lambda_2(\theta)\}$ are singular points of $J_3(S)$.

Proof.—It is easy to see that under the conditions imposed upon $\phi_1(z)$ and $\phi_2(z)$ we may write

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{\log|\phi_1(z)\phi_2(z)|}{r} = \lambda_1(\theta) + \lambda_2(\theta) \; |\theta| < \alpha < \frac{\Pi}{2}. \quad (4)$$

Hence by virtue of a theorem of Prof. Srivastava[4] the theorem follows at once.

A similar result holds good also for the functions $H_1(S)$, $H_2(S)$ and $H_3(S)$ defined by (2.3), (2.4) and (2.2) respectively in Theorem (3) when the condition $|\theta| \leqslant \alpha$, $\alpha \geqslant \frac{\Pi}{2}$ is replaced by $|\theta| \leqslant \alpha < \frac{\Pi}{2}$.

It is well known that if $h_1(S) = \Sigma a_n e^{-sn}$ and $h_2(S) = \Sigma b_n e^{-sn}$ have one singularity each, then it is not necessary that $H(S) = \Sigma a_n b_n e^{-sn}$ should also have one singularity. In the case of the three integrals $J_1(S)$, $J_2(S)$ and $J_3(S)$ we prove the following theroem.

Theorem 5.—If

$$J_1(S) = \int_0^\infty \phi_1(z)e^{-Sz}\,dz$$

has got singularity at $S = A_1 - iB_1$. and if

$$J_2(S) = \int_0^\infty \phi_2(z)\, e^{-Sz}\, dz$$

has got singularity at $S = A_2 - iB_2$. then

$$J_3(S) = \int_0^\infty \phi_1(z)\, \phi_2(z)\, e^{-Sz}\, dz$$

has got singularity at $A_1 + A_2 - iB_1 - B_2$.

Proof.—The necessary and sufficient condition that an integral function $\phi(z)$ should satisfy the asymptotic equality

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{\log |\phi(re^{i\theta})|}{r} = A\cos\theta + B\operatorname{Sin}\theta \quad (\text{for } |\theta| \leqslant \Pi),$$

is that $\phi(z)$ should be of the form

$$\phi(z) = \frac{1}{2\Pi i} \int_{c'} e^{uz}\, d(u)\, du \tag{3.1}$$

where c' is a simple contour enclosing the point $u = A - iB$, and $d(u)$ is an integral function of $\frac{1}{u-(A-iB)}$ not identically a constant.[5]

Also Polya[6] has proved that if $\phi(z)$ be an integral function of z, it can be expressed in the form

$$\phi(z) = \frac{1}{2\pi i} \int_{c''} e^{uz}\, J(u)\, du \tag{3.2}$$

where

$$J(S) = \int_0^\infty \phi(z)\, e^{-Sz}\, dz$$

and c'' is any contour lying in the region of regularity of $J(u)$.

We may therefore write by (3.1) and (3.2) that

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{\log |\phi_1(z)|}{r} = A_1\cos\theta + B_1\operatorname{Sin}\theta, \quad |\theta| \leqslant \pi, \tag{3.3}$$

and

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{\log |\phi_2(z)|}{r} = A_2\cos\theta + B_2\operatorname{Sin}\theta \quad |\theta| \leqslant \pi \tag{3.4}$$

Now the equalities (3.3) and (3.4) at once lead to the inequality

$$|\phi_1(z)\, \phi_2(z) < e^{\{\lambda_1(\phi)+\lambda_2(\phi)+\epsilon\}r}$$

where

$$\lambda_1(\theta)+\lambda_2(\theta)=A_1+A_2\cos\theta+\overline{B_1+B_2}\sin\theta.$$

Also we have

$$\phi_1(z)\,\phi_2(z)=0\,(e^{Ar})$$

where A is some real constant.

Hence by virtue of a theorem of Prof. Srivastava[7] we can write

$$\operatorname*{Lim\,Sup}_{r\to\infty}\frac{\log|\phi_1(z)\,\phi_2(z)|}{r}=A_1+A_2\cos\psi+B_1+B_2\sin\psi_1$$

$$(\text{for } |\psi|\leqslant\pi),$$

and by the same theorem

$$J_3(S)=\int_0^\infty\phi_1(z)\,\phi_2(z)\,e^{-Sz}\,dz$$

has got the sole singularity at $S=(\overline{A_1+A_2}-i\overline{B_1+B_2})$.

The immediate consequences of Theorem 5 are the following two theorems.

Theorem 6.—If in Theorem 3 we take $\phi_1(z)$ and $\phi_2(z)$ to be integral functions of z satisfying the conditions of the theorem everywhere in the z-plane then if

$$H_1(S)=\Sigma\,\phi_1(\lambda_n)\,\lambda'_n\,e^{-S\lambda_n}$$

has got singularity at A_1-iB_1 and

$$H_2(S)=\Sigma\,\phi_2(\lambda_n)\,\lambda'_n\,e^{-S\lambda_n}$$

has got singularity at A_2-iB_2, then

$$H_3(S)=\Sigma\,\phi_1(\lambda_n)\,\phi_2(\lambda_n)\,e^{-S\lambda_n}\,\lambda'_n$$

has got the sole singularity at $S=\overline{A_1+A_2}-i\overline{B_1+B_2}$.

Theorem 7.—If $\phi_1(z)$ and $\phi_2(z)$ are integral functions of z satisfying the conditions $\phi_1(z)=0\,(e^{k_1r})$ and $\phi_2(z)=0\,(e^{k_2r})$, everywhere in the plane, and if

$$F_1(S)=\Sigma\,\phi_1(n)\,e^{-Sn}\qquad |t|\leqslant\Pi$$

has got singularity at $S=A_1-iB_1$, and

$$F_2(S)=\Sigma\,\phi_2(n)\,e^{-Sn},\qquad |t|\leqslant\Pi$$

4

has got singularity at $S = A_2 - iB_2$, then, if $(B_1 + B_2) < \Pi$, the only singular point of

$$F_3(S) = \Sigma \phi_1(n) \phi_2(n) e^{-Sn}$$

is $S = A_1 + A_2 - iB_1 + B_2$.

I am very much indebted to Prof. P. L. Srivastava for his many useful suggestions.

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ON THE CONVERGENCE OF GENERALISED LAPLACESTIELTJES INTEGRALS

By Snehlata

(Department of Mathematics, Allahabad University)

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1. In this paper it is proposed to examine the existence and the convergence of the integral

$$f(s)=\int_0^\infty (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t), \tag{1.1}$$

and also to establish a relation between the order property of $\alpha(t)$ and the convergence property of the corresponding generalised Laplacestieltjes integral.

Let the real and the imaginary parts of the complex variable s be σ and τ respectively so that

$$s=\sigma+i\tau$$

and let $\alpha(t)$ be a function of the real variable t in the interval $0\leqslant t\leqslant\infty$ and be of bounded variation in $0\leqslant t\leqslant R$ for every +ve R.

If $\alpha(t)$ be of bounded variation in the interval $\epsilon\leqslant t\leqslant R$ for every +ve ϵ and every +ve R, one may use the notation

$$\int_{0+}^\infty (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t)$$

$$=\operatorname*{Lt}_{\substack{\epsilon\to 0\\ R\to\infty}} \int_\epsilon^R (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t)$$

2. Theorem:—If

$$\operatorname*{u.b.}_{0<u<\infty} \left| \int_0^u (qs_0t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)s_0t}\, W_{l,n}(qs_0t)\, d\alpha(t) \right| = M < \infty, \tag{2.1}$$

then the integral (1.1) converges for every $s=(\sigma+i\tau)$ for which $\sigma>\sigma_0$ and

$$\int_0^\infty (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t) = \int_0^\infty \frac{K(t)}{m(t)}\, \beta(t)\, dt, \tag{2.2}$$

where

$$K(t) = \Big[(qst)^{c-3/2} e^{-(p-\frac{1}{2}q)st} \, W_{l,n}(qst) \{ qs(l - \tfrac{1}{2} + qst)$$

$$- \tfrac{1}{2} qst \left(\left(p - \frac{q}{2} \right) s + (qs)^2 \right) \} - (qs)^2 \{ n^2 - (l - \tfrac{1}{2})^2 \} (qst)^{c-3/2}$$

$$\times e^{-(p-\frac{1}{2}q)st} \, W_{l-1,n}(qst) \Big] (qs_0 t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l,n}(qs_0 t)$$

$$- \Big[(qs_0 t)^{c-3/2} e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l,n}(qs_0 t) \{ qs_0 (c - \tfrac{1}{2} + qs_0 t)$$

$$- \tfrac{1}{2} qs_0 t \left(\left(p - \frac{q}{2} \right) s_0 - (qs_0)^2 \right) \}$$

$$- (qs_0)^2 \{ n^2 - (l - \tfrac{1}{2})^2 \} (qs_0 t)^{c-3/2}$$

$$\times e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l-1,n}(qs_0 t) \Big] (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q) st} \, W_{l,n}(qst),$$

$$m(t) = \{ (qs_0 t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l,n}(qs_0 t) \}^2$$

and

$$\beta(u) = \int_0^u (qs_0 t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l,n}(qs_0 t) \, d\alpha(t) \tag{2.3}$$

the integral on the right-hand side of (2.3) converges absolutely.

For, we have, by virtue of (2.3)

$$\int_0^R (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st} \, W_{l,n}(qst) \, d\alpha(t)$$

$$= \int_0^R \frac{(qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st} \, W_{l,n}(qst)}{(qs_0 t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)s_0 t} \, W_{l,n}(qs_0 t)} \, d\beta(t)$$

$$= \frac{(qsR)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)sR} \, W_{l,n}(qsR)}{(qs_0 R)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)s_0 R} \, W_{l,n}(qs_0 R)} - \int_0^R \frac{S \dfrac{d}{dt}(Q) - Q \dfrac{d}{dt}(S)}{S^2} \beta(t) \, dt,$$

where

$$Q = (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st} \, W_{l,n}(qst)$$

and

$$S = (qs_0 t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q) s_0 t} \, W_{l,n}(qs_0 t)$$

The integrated portion vanishes in the limit when $R \to \infty$ since

$$W_{l,n}(z) = 0(e^{-\frac{1}{2}z} z^l), \text{ for } z \text{ large.}$$

If we simplify the integrand with the help of

$$zW'_{k,m}(z) = (k - \tfrac{1}{2}z)\, W_{k,m}(z) - \{m^2 - (k - \tfrac{1}{2})^2\}\, W_{k-1,m}(z) \qquad \text{(A)}$$

we arrive at once at the theorem.

It is easy to see from this theorem that if the integral (1.2) converges for $s_0 = \sigma_0 + i\tau_0$, it converges for all $s = \sigma + i\tau$ for which $\sigma > \sigma_0$.

Hence the region of convergence for (1.1) is a half-plane and if there be a number σ_c such that the integral converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$, we call it to be the abscissa of convergence of the integral and $\sigma = \sigma_c$ as the axis of convergence.

3. Now, the relation between the order property of $\alpha(t)$ and the convergence property of the corresponding generalised Laplacestieltjes integral will be established.

Theorem.—If

$$\alpha(t) = 0\,(e^{\gamma t}), \quad (t \to \infty)$$

for some real number γ, then the integral (1.2) converges for $p\sigma > \gamma$.

For, integrating by parts, we have

$$\int_0^\infty (qst)^{c-\frac{1}{2}}\, e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t)$$

$$= \left[\alpha(t)\,(qst)^{c-\frac{1}{2}}\, e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\right]_0^R$$

$$- \int_0^R \alpha(t) \frac{d}{dt} \left[(qst)^{c-\frac{1}{2}}\, e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\right] dt$$

$$= \left[\alpha(R)\,(qsR)^{c-\frac{1}{2}}\, e^{-(p-\frac{1}{2}q)sR}\, W_{l,n}(qsR)\right]$$

$$- \int_0^R \alpha(t) \left[(qst)^{c-3/2}\, e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, \{qs\,(c - \tfrac{1}{2} + qsl)\right.$$

$$- \tfrac{1}{2}\, qst\, \big((p - \tfrac{1}{2}q)s + (qs)^2\big)\} - (qs)^2\, \{n^2 -$$

$$\left.(l - \tfrac{1}{2})^2\}\, (qst)^{c-3/2}\, e^{-(p-\frac{1}{2}q)st}\, W_{l-1,n}(qst)\right] dt,$$

using the formula (A) (Whittaker and Watson, p. 352, Q. 3).

Now

$$W_{l,n}(x) = 0\,(x^l e^{-\frac{1}{2}x}) \text{ when } x \text{ is large.}$$

Therefore

$$\alpha(\mathrm{R})\,(qs\mathrm{R})^{c-\frac{1}{2}}\,e^{-(p-\frac{1}{2}q)s\mathrm{R}}\,\mathrm{W}_{l,n}(qs\mathrm{R})$$

$$= ke^{\gamma\mathrm{R}}(qs\mathrm{R})^{c-\frac{1}{2}+l}\,e^{-ps\mathrm{R}} \qquad \text{as } \mathrm{R}\to\infty$$

$$= k\,(qs\mathrm{R})^{c+l-\frac{1}{2}}\,e^{(\gamma-ps)\mathrm{R}} \qquad \text{as } \mathrm{R}\to\infty$$

$$= 0\,(1) \text{ if } \mathrm{R}\,(ps) > \gamma \;\text{ or if } p\sigma > \gamma,$$

where σ is the real part of s.

Since $\alpha(t)$ is assumed to be of bounded variation in every finite interval, our hypothesis implies that

$$\alpha(t) = 0\,(e^{\gamma t})$$

i.e.,

$$\operatorname*{Lt}_{t\to\infty}\frac{\alpha(t)}{e^{\gamma t}} = \mathrm{M}$$

$$\therefore\quad |\,\alpha(t)\,| \leqslant \mathrm{M}\,e^{\gamma t}, \qquad \text{for } 0 \leqslant t < \infty,$$

where M is some constant.

Hence

$$\int_0^\infty (qst)^{c-\frac{1}{2}}\,e^{-(p-\frac{1}{2}q)st}\,\mathrm{W}_{l,n}(qst)\,\alpha(t)\,dt < \mathrm{M}\int_0^\infty [(qs)^2\{n^2-(l-\tfrac{1}{2})^2\}(qst)^{c-3/2}$$

$$e^{-(p-\frac{1}{2}q)st}\,\mathrm{W}_{l-1,n}(qst) - (qst)^{c-3/2}\,e^{-(p-\frac{1}{2}q)st}\{(qs)(l-\tfrac{1}{2}+qsl) - \tfrac{1}{2}qst$$

$$\times\,[(p-\tfrac{1}{2}q)s+(qs)^2]\,\mathrm{W}_{l,n}(qst)\}]\,e^{\gamma t}\,dt$$

$$= \mathrm{M}\int_0^\infty [(qs)^2\{n^2-(l-\tfrac{1}{2})^2\}(qst)^{c-3/2}\,e^{-(p-\frac{1}{2}q-\frac{\gamma}{s})st}\,\mathrm{W}_{l-1,n}(qst)$$

$$-(qs)\,(c-\tfrac{1}{2}+qsl)\,(qst)^{c-3/2}\,.\,e^{-(p-\frac{1}{2}q-\frac{\gamma}{s})st}\,\mathrm{W}_{l,n}(qst)$$

$$+\tfrac{1}{2}\{(p-\tfrac{1}{2}q)s+(qs)^2\}\,(qst)^{c-\frac{1}{2}}\,e^{-(p-\frac{1}{2}q-\frac{\gamma}{s})st}\,\mathrm{W}_{l,n}(qst)]\,dt,$$

$$< \mathrm{M}\left[(qs)^{c-\frac{1}{2}}\{n^2-(l-\tfrac{1}{2})^2\}\,\frac{\Gamma(c+n)\,\Gamma(c-n)}{\Gamma(c-l+3/2)\cdot(qs)^{c-\frac{1}{2}}}\right.$$

$$\times\;{}_2\mathrm{F}_1\left\{\begin{matrix}c+n,\,c-n\\ c-l+3/2\end{matrix}\;;\,1+\frac{r}{qs}-\frac{p}{q}\right\} - (qs)^{c-\frac{1}{2}}\,(c-\tfrac{1}{2}+qsl)$$

$$\times\;\frac{\Gamma(c+n)\,\Gamma(c-n)}{\Gamma(c-l+\frac{1}{2})\,(qs)^{c-\frac{1}{2}}}\cdot{}_2\mathrm{F}_1\left\{\begin{matrix}c+n,\;c-n\\ c-l+\frac{1}{2}\end{matrix}\;;\,1+\frac{\gamma}{qs}-\frac{p}{q}\right\}$$

$$+ \tfrac{1}{2}\{(p - \tfrac{1}{2}q)s + (qs)^2\}(qs)^{c-\frac{1}{2}} \frac{\Gamma(c+n+1)\,\Gamma(c-n+1)}{\Gamma(c-l+3/2)\,(qs)^{c+\frac{1}{2}}}$$

$$\times {}_2F_1 \left\{ \begin{matrix} c+n+1,\ c-n+1 \\ c-l+3/2 \end{matrix} ; 1 + \frac{\gamma}{qs} - \frac{p}{q} \right\} \Big], \tag{3.1}$$

provided that $R(c \pm n) > 0$, $R(ps) = p\sigma > \gamma$ integrating the right-hand side with the help of Goldstein's integral

$$\int_0^\infty x^{l-1} e^{-(a^2+\frac{1}{2})x}\, W_{k,m}(x)\, dx = \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}$$

$$\times {}_2F_1 \left[\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ l-k+1 \end{matrix} ; -a^2 \right],$$

where

$$R(l \pm m + \tfrac{1}{2}) > 0,\ R(a^2+1) > 0.$$

The right-hand side is a convergent series if $\gamma < p.R(s) < p\sigma$ and so the integral on the left hand side of the inequality converges absolutely for $p\sigma > \gamma$.

Thus

$$\int_0^\infty (qst)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\, d\alpha(t)$$

$$= \int_0^\infty (qs)^2 \{n^2 - (l-\tfrac{1}{2})^2\}(qst)^{c-3/2} e^{-(p-\frac{1}{2}q)st}\, W_{l-1,n}(qst)$$

$$- (qst)^{c-3/2} e^{-(p-\frac{1}{2}q)st}\, W_{l,n}(qst)\{qs(c - \tfrac{1}{2} + qsl)$$

$$- \tfrac{1}{2} qst\left((p - \tfrac{1}{2}q)s + (qs)^2\right)\}]\, \alpha(t)\, dt$$

and our theorem is established.

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AN EXTENSION OF HADAMARD'S MULTIPLICATION THEOREM

By Nirmala Pandey

Given any factorial series of the form

$$\Omega'(Z) = \sum_{n=0}^{\infty} \frac{e^{Ain} g'(n)}{Z(Z+1)\ldots(Z+n)}. \qquad (0 < A < 2\pi) \tag{1}$$

whose abscissa λ of convergence is finite, it is known[1] that if the function $g'(n)$ occurring in the coefficient of this series is such that when considered as a function $g'(\omega)$ of the complex variable $\omega = x + iy$, (*a*) it is single valued and analytic throughout all portions of the ω-plane lying to the right of (or upon) the vertical line $\omega = -\frac{1}{2} + iy$ and (*b*) there corresponds to any arbitrarily small positive number ϵ, a positive constant K_ϵ such that for all values of $x \geqslant -\frac{1}{2}$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g'(x+iy)}{g'(x)} \right| < K_\epsilon \, e^{(A-\frac{\pi}{2}-\epsilon)y}$$

and

$$\left| \frac{g'(x-iy)}{g'(x)} \right| < K_\epsilon \, e^{(2\pi-A-\frac{\pi}{2}-\epsilon)y}$$

and moreover

(*c*) there exists a positive constant q, dependent on ϵ but independent of Z, such that when Z is confined to any finite region, we have

$$\left| \frac{g'(x)}{\Gamma(Z+x+1)} \right| < \frac{N}{x^{1+\epsilon}}$$

where $x > q$ and where N is a positive constant, then the function $\Omega(Z)$ defined by the series (1) when $R(Z) > \lambda$ may be extended analytically throughout the whole finite Z-plane except in the neighbourhoods of the points $Z = 0, -1, -2, \ldots\ldots$

Let us suppose then that

$$f(Z)=\sum_{n=0}^{\infty}\frac{e^{Ain}\, g_1(n)}{Z(Z+1)\ldots.(Z+n)}$$

is analytic throughout the whole finite Z-plane except in the neighbourhoods of the points $Z=0, -1, -2, \ldots.$

and that

$$g(Z)=\sum_{n=0}^{\infty}\frac{e^{Ain}\, g_2(n)}{Z(Z+1)\ldots.(Z+n)}$$

is also analytic throughout the whole finite Z-plane except in the neighbourhoods of the points $Z=0, -1, -2, \ldots..$

Then what can be said about the singularities of the function

$$F(Z)=\sum_{n=0}^{\infty}\frac{e^{2Ain}\, g_1(n)\, g_2(n)}{Z(Z+1)\ldots.(Z+n)}$$

whose coefficients are the products of those in the given series and $g_1(n)$ and $g_2(n)$ are similar to $g'(n)$ as defined by (*a*), (*b*) and (*c*).

My object in this paper is to find the singularities of this product series. Writing $g(n)$ for $\{g_1(n)\times g_2(n)\}$ I wish to prove the following theorem:—

Let the factorial series

$$\Omega(Z)=\sum_{n=0}^{\infty}\frac{e^{2Ain}\, g(n)}{Z(Z+1)\ldots.(Z+n)},\quad (0<A<2\pi) \tag{2.1}$$

have a finite abscissa of convergence equal to λ, $g(\omega)$ $(\omega=x+iy)$ be such that

(2.2) in the half plane $R(\omega)>-\frac{1}{2}$, $g(\omega)$ is single-valued and analytic
(2.3) for all values of ω for which $x>-\frac{1}{2}$ and y is sufficiently large we may write

$$\left|\frac{g(x+iy)}{g(x)}\right| < K_\epsilon\ e^{(2A-\frac{\pi}{2}-\epsilon)\,y}$$

$$\left|\frac{g(x-iy)}{g(x)}\right| < K'_\epsilon\, e^{(\frac{7\pi-2A-\epsilon}{2})\,y}$$

(**K_ϵ and K'_ϵ being constants**)

Moreover let there exist a positive constant q, dependent on ϵ but independent of Z, such that when Z is confined to any finite region, we have

$$\left|\frac{g(x)}{\Gamma(Z+x+1)}\right| < \frac{N}{x^{1+\epsilon}}, \tag{2.4}$$

where $x > q$ and where N is a positive constant then the function $\Omega(Z)$ defined by the series (2.1) can be extended analytically throughout the whole finite Z-plane, except possibly in the neighbourhoods of the points $Z = 0, -1, -2, \ldots$ and throughout this region can be defined by the equation

$$\Omega(Z) = \Gamma(Z) \int_{-1/2}^{\infty} \frac{e^{2Aix-2\pi ix}\, g(x)}{\Gamma(Z+x+1)}\, dx$$

$$+ 2\pi i \Gamma(Z) \int_{+\infty}^{-\infty} \frac{\displaystyle\int_0^y \frac{e^{2\pi(S-y)}\, g(-\frac{1}{2}+iS)\, e^{2Ai(-\frac{1}{2}+iS)}}{\Gamma(Z+\frac{1}{2}+iS)}\, ds}{(e^{-2\pi y}+1)^2}\, dy \tag{2.5}$$

The proof of this theorem is based upon Cauchy's integral theorem in the calculus of Residues and we can write

$$\sum_{n=0}^{n} \frac{e^{2Ain}\, g(n)\, \Gamma(Z)}{\Gamma(Z+n+1)} = 2\pi i\, \Gamma(Z) \int_{C_n} \frac{e^{2\pi i\omega}\, I(Z, \omega)}{(e^{2\pi i\omega}-1)^2}\, d\omega, \tag{2.6}$$

where

$$I(\omega, Z) = \int_0^{\omega} \frac{e^{-2\pi i\omega + 2Ai\omega}\, g(\omega)}{\Gamma(Z+\omega+1)}\, d\omega \tag{2.7}$$

and where the path of integration C_n is the rectangle formed by the lines (1) $\omega = x + ip$ (2) $\omega = -\frac{1}{2} + iy$ (3) $\omega = x - ip$ and (4) $\omega = n + \frac{1}{2} + iy$. Let us denote the integrals along these sides by A, B, D and E respectively.

In order to evaluate the contour integral

$$\int_{C_n} \frac{e^{2\pi i\omega}\, I(Z, \omega)}{(e^{2\pi i\omega}-1)^2}\, d\omega \tag{2.8}$$

we may take the path of integration in (2.7) along the x-axis from $\omega = 0$ to $\omega = x$ and then along a line parallel to the y-axis to the point $\omega = x + iy$.

The function $I(\omega, Z)$ can then be written in the form

$$I(\omega, Z) = R(x, Z) + iS(x, y, z), \tag{2.9}$$

where

$$R(x, Z) = \int_0^x \frac{\exp.(-2\pi ix)\, g(x)\, e^{2\lambda ix}}{\Gamma(Z + x + 1)}\, dx \tag{2.10}$$

and

$$S(x, y, Z) = \int_0^y \frac{\exp.\{-2\pi i(x + iy)\}\, g(x + iy)\, e^{2\lambda i(x+iy)}}{\Gamma(Z + x + 1 + iy)}\, dy. \tag{2.11}$$

Consider first the contribution A. To evaluate A we consider the integral

$$\int_{n+1/2}^{-1/2} \frac{\exp.\, 2\pi i(x + ip)\,[R(x, Z) + iS(x, p, Z)]}{(e^{2\pi i(x+ip)} - 1)^2}\, dx$$

we can evidently write

$$|A| \leqslant \left|\int_{n+1/2}^{-1/2} \frac{R(x, Z)\, dx}{\exp.(2\pi p)}\right| + \left|\int_{n+1/2}^{-1/2} \frac{S(x, y, Z)}{\exp.(2\pi p)}\, dx\right| \tag{2.12}$$

$R(x, Z)$ being bounded when x is bounded, the first integral on the right of (2.12) approaches zero as p approaches infinity.

Also the absolute value of the integrand in the second integral in question on the right of (2.12) is of the order of $\exp.(-\frac{1}{2}\epsilon p)$ for p large and since ϵ is positive we conclude that

$$\lim_{p \to \infty} A = 0$$

We can in a similar manner prove that

$$\lim_{p \to \infty} D = 0$$

To consider next the contribution E we consider

$$E_1 = -\left[i R\left(n + \tfrac{1}{2}, Z\right) \int_{-\infty}^{\infty} \frac{\exp.(-2\pi y)}{(e^{-2\pi y} + 1)^2}\, dy \right.$$

$$\left. - \int_{-\infty}^{\infty} \frac{\exp.(-2\pi y)\, S\left(n + \tfrac{1}{2}, y, Z\right)}{(e^{-2\pi y} + 1)^2}\, dy \right] \tag{2.13}$$

Evaluating the first integral on the right of (2.13) we have

$$\int_{-\infty}^{\infty} \frac{\exp.(-2\pi y)}{(e^{-2\pi y} + 1)^2}\, dy = \frac{1}{2\pi} \int_{0}^{\infty} \frac{dt}{(t+1)^2} = \frac{1}{2\pi}.$$

Considering the second integral on the right of (2.13) we can write it in the form

$$\int_{-\infty}^{\infty} \frac{\exp.(-2\pi y)\, S\left(n + \tfrac{1}{2}, y, Z\right)}{(e^{-2\pi y} + 1)^2}\, dy$$

$$= \frac{g(n)}{\Gamma(Z + n + 1)} \int_{-\infty}^{\infty} \frac{\exp.(-2\pi y)\, S_1\left(n + \tfrac{1}{2}, y, Z\right)}{(e^{-2\pi y} + 1)^2}\, dy \tag{2.14}$$

where

$$S_1\left(n + \tfrac{1}{2}, y, Z\right) = -e^{2Ai\left(n + \frac{1}{2}\right)} \int_{0}^{y} \frac{\Gamma(Z + n + 1)\, g\left(n + \tfrac{1}{2} + iy\right) e^{2\pi y - 2Ay}}{\Gamma(Z + n + 3/2 + iy)\, g(n)}\, dy$$

Since from the known property of the Gamma function we have

$$\left| \frac{\Gamma(Z + n + 1)}{\Gamma\left(Z + n + \frac{3}{2} + iy\right)} \right| < \frac{\left| Z + n + 1 \right|^{Z+n+1}}{\left| Z + n + \frac{3}{2} + iy \right|^{Z+n+1}}\, e^{y \tan^{-1} \frac{y}{Z + n + \frac{3}{2}}} \leqslant e^{\frac{\pi}{2}|y|}$$

We find that if we take into account the behaviour of $(e^{-2\pi y} + 1)^2$ for $|y|$ large, the integrand of the right-hand member of (2.14) is of the order of $\exp.(-\epsilon |y|)$ for $|y|$ large. The integral in question therefore converges. Moreover, since the series (2.1) was assumed convergent, we have

$$\lim_{n \to \infty} \frac{g(n)}{\Gamma(Z + n + 1)} = 0$$

Consequently the integral on the left of (2.14) tends to zero as n approaches infinity. The contribution E is therefore given by

$$E = \Gamma(Z) \lim_{n\to\infty} R\left(n + \tfrac{1}{2}, Z\right) \tag{2.15}$$

We next consider the contribution B arising from the integration along the side where $\omega = -\frac{1}{2} + iy$. To evaluate B we consider the integrals

$$-\Gamma(Z)\, 2\pi i \int_{\infty}^{-\infty} i\, \frac{\exp.(-2\pi y)\, R\left(-\frac{1}{2}, Z\right)}{(e^{-2\pi y} + 1)^2}\, dy$$

$$+\Gamma(Z)\, 2\pi i \int_{\infty}^{-\infty} \frac{\exp.(-2\pi y)\, S\left(-\frac{1}{2}, y, Z\right)}{(e^{-2\pi y} + 1)^2}\, dy \tag{2.16}$$

The first of these integrals, when combined with the contribution E can be written in the form

$$\Gamma(Z) \lim_{n\to\infty} \left[R\left(n + \tfrac{1}{2}, Z\right) - R\left(-\tfrac{1}{2}, Z\right)\right]$$

$$= \Gamma(Z) \int_{-1/2}^{\infty} \frac{\exp.(-2\pi i x)\, g(x)\, e^{2Aix}}{\Gamma(Z + x + 1)}\, dx$$

$$= \Gamma(Z) \int_{-1/2}^{\infty} \frac{e^{2Aix - 2\pi i x}\, g(x)}{\Gamma(Z + x + 1)}\, dx \tag{2.17}$$

Finally we investigate the second integral in (2.16). It may be written in the form

$$2\pi i\, \Gamma(z) \int_{\infty}^{-\infty} \frac{\displaystyle\int_{0}^{y} \frac{e^{2\pi(S-y)}\, g\left(-\frac{1}{2} + iS\right) e^{2Ai\left(-\frac{1}{2}+iS\right)}}{\Gamma\left(z + \frac{1}{2} + iS\right)}\, ds}{(e^{-2\pi y} + 1)^2}\, dy \tag{2.18}$$

The value of the right-hand member of (2.6) is therefore the sum of the expressions (2.17) and (2.18). We therefore obtain equation (2.5).

We have up to this point restricted z to values which are real, positive and greater than λ, the abscissa of convergence. The function $\Omega(z)$ defined by the series (2.1) is analytic throughout the half-plane $R(z) > \lambda$ with the exception of the points $z = 0, -1, -2, \ldots$. The right side of (2.5) enables us to continue $\Omega(z)$ analytically over any finite region of the z-plane

which does not contain the points $z = 0, -1, -2, \ldots\ldots$ This is easily proved by the uniform convergence of the two integrals involved in (2.5). The first integral converges uniformly since it is dominated by $\int^{\infty} x^{-1-\epsilon}\, dx$ and the second converges uniformly since it is dominated by $\int^{\infty} \exp. (-\epsilon y)\, dy$. The theorem is therefore established.

We thus obtain the following theorem:

Theorem.—If

$$f(z) = \sum_{n=0}^{\infty} \frac{e^{Ain}\, g_1(n)}{z(z+1)\ldots.(z+n)} \tag{2.19}$$

has got singularities at $z = 0, -1, -2$.

and if

$$g(z) = \sum_{n=0}^{\infty} \frac{e^{Ain}\, g_2(n)}{z(z+1)\ldots.(z+n)} \tag{2.20}$$

has got singularities at $z = 0, -1, -2$.

then

$$\mathrm{F}(z) = \sum_{n=0}^{\infty} \frac{e^{2Ain}\, g_1(n)\, g_2(n)}{z(z+1)\ldots.(z+n)} \tag{2.21}$$

has also got singularities at $z = 0, -1, -2$.

REMARKS

The coefficients $g_1(n), g_2(n)$ as contemplated in the theorem may be $\frac{1}{\Gamma(n+c_1)}$ and $\frac{1}{\Gamma(n+c_2)}$. It is easy to see that $\frac{1}{\Gamma(n+c_1)\,\Gamma(n+c_2)}$ satisfies the condition (2.3).

It is easy to see that the above theorem is not always true. The function

$$f_1(z) = \sum_{n=0}^{\infty} \frac{e^{Ain}\, \Gamma(n+c_1)}{z(z+1)\ldots.(z+1)}$$

has got singularities at $z = 0, -1, -2, \ldots$ and the function

$$f_2(z) = \sum_{n=0}^{\infty} \frac{e^{\lambda i n} \, \Gamma(n + c_2)}{z(z+1) \ldots (z+1)}$$

has got singularities at $z = 0, -1, -2, \ldots$ but we cannot say anything about the singularities of

$$F_1(z) = \sum_{n=0}^{\infty} \frac{e^{2\lambda i n} \, \Gamma(n + c_1) \, \Gamma(n + c_2)}{x(z+1) \ldots (z+n)} .$$

I wish to thank Prof. P. L. Srivastava for his guidance and help in the preparation of this note.

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[1] *American Journal of Math.*, 1931, **53**, 771.

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